# Logarithmic Variance for the Height Function of Square Ice

Gourab Ray

Joint work with: H. Duminil-Copin (IHES, UniGe), M. Harel (Tel Aviv), B. Laslier (Paris–Diderot), A. Rauofi (ETH)

University of Victoria

November 13, 2019

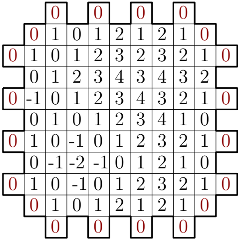
# Uniform Homomorphisms

Let $\Lambda$ be a finite subgraph of $\mathbb{Z}^2$. Consider $h : V(\Lambda) \to \mathbb{Z}$ such that, for any neighboring vertices $u$ and $v$, $|h_u - h_v| = 1$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 0 | | 0 | | 0 | | 0 | | |
| | 0 | 1 | 0 | 1 | 2 | 1 | 2 | 1 | 0 | |
| 0 | 1 | 0 | 1 | 2 | 3 | 2 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 2 | 3 | 4 | 3 | 4 | 3 | 2 | |
| 0 | -1 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 3 | 4 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | -1 | -2 | -1 | 0 | 1 | 2 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 1 | 2 | 1 | 0 | |
| | | 0 | | 0 | | 0 | | 0 | | |

# Uniform Homomorphisms

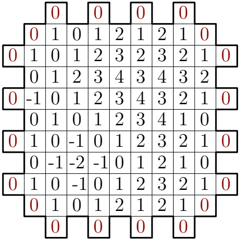
Let $\Lambda$ be a finite subgraph of $\mathbb{Z}^2$. Consider $h : V(\Lambda) \to \mathbb{Z}$ such that, for any neighboring vertices $u$ and $v$, $|h_u - h_v| = 1$.




We will take $\Lambda_n$ to be an (even) square of side length $2n$, with $h \equiv 0$ on the boundary; uniformly pick one such function $h$ and call this measure $\phi^0_{\Lambda_n}$.

# Uniform Homomorphisms

Let $\Lambda$ be a finite subgraph of $\mathbb{Z}^2$. Consider $h : V(\Lambda) \to \mathbb{Z}$ such that, for any neighboring vertices $u$ and $v$, $|h_u - h_v| = 1$.




We will take $\Lambda_n$ to be an (even) square of side length $2n$, with $h \equiv 0$ on the boundary; uniformly pick one such function $h$ and call this measure $\phi^0_{\Lambda_n}$.
How does $\mathrm{Var}(h_0)$ behave as $n \to \infty$?

# Random Surfaces

In one dimension, this model is simply a simple random walk which is classical, with variance of order $n$.

# Random Surfaces

In one dimension, this model is simply a simple random walk which is classical, with variance of order $n$.

In $\mathbb{Z}^2$, the uniform homomorphism model is conjectured to be one of the (many) random-surface models that can have one of two behaviors:

# Random Surfaces

In one dimension, this model is simply a simple random walk which is classical, with variance of order $n$.

In $\mathbb{Z}^2$, the uniform homomorphism model is conjectured to be one of the (many) random-surface models that can have one of two behaviors:

- $\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr}$, for some $k > 0$ **(localized)** , or

# Random Surfaces

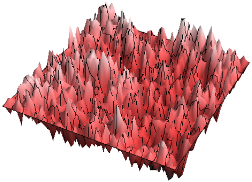
In one dimension, this model is simply a simple random walk which is classical, with variance of order $n$.

In $\mathbb{Z}^2$, the uniform homomorphism model is conjectured to be one of the (many) random-surface models that can have one of two behaviors:

- $\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr}$, for some $k > 0$ **(localized)** , or
- $k \log n \leq \phi^0_{\Lambda_n}[h_0^2] \leq K \log n$ for some $k, K > 0$. **(delocalized)**

## Scaling limit

In the delocalized phase, the model is supposed to behave like a Gaussian free field in the scaling limit which is conformally invariant.



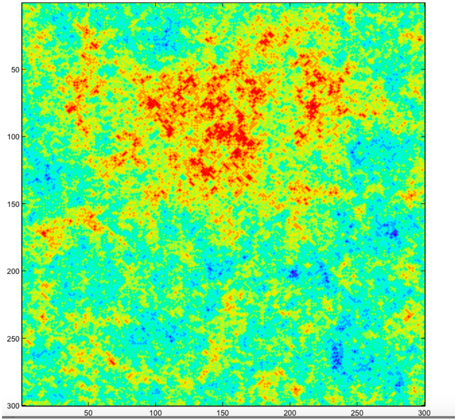



Figure: Left: Due to Scott Sheffield, Right: Due to Ron Peled

# Main theorem

**Theorem (DCHLRR, 19)**

*For the uniform homomorphism model,* $\exists c, C > 0$ *so that for all* $n \geq 1$,

$$c \log n \leq Var_{\Lambda_n^0}(h_0) \leq C \log n.$$

# Dichotomy Theorem

Our strategy is to prove the following Dichotomy theorem:

# Dichotomy Theorem

Our strategy is to prove the following Dichotomy theorem:

**Theorem (DCHLRR, 18)**

For the uniform homomorphism, either:

# Dichotomy Theorem

Our strategy is to prove the following Dichotomy theorem:

**Theorem (DCHLRR, 18)**

For the uniform homomorphism, either:

- $\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr^\alpha}$, for some $k, \alpha > 0$, or

# Dichotomy Theorem

Our strategy is to prove the following Dichotomy theorem:

**Theorem (DCHLRR, 18)**

For the uniform homomorphism, either:

- $\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr^\alpha}$, for some $k, \alpha > 0$, or
- $\exists c, C > 0$ so that for all $n \geq 1$,

$$c \log n \leq \mathrm{Var}(h_0) \leq C \log n.$$

# Dichotomy Theorem

Our strategy is to prove the following Dichotomy theorem:

**Theorem (DCHLRR, 18)**

For the uniform homomorphism, either:

- ~~$\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr^\alpha}$, for some $k, \alpha > 0$~~ [Chandgotia, Peled, Sheffield, Tassy ['19]]
- $\exists c, C > 0$ so that for all $n \geq 1$,

$$c \log n \leq \mathrm{Var}(h_0) \leq C \log n.$$

# Perspectives

## History and perspectives

One can view this model from different (not necessarily disjoint) perspectives and flavours.

- A a model of random graph homomorphism between two graphs and vary the graphs (gets into computer science questions like graph colorings).

# Perspectives

## History and perspectives

One can view this model from different (not necessarily disjoint) perspectives and flavours.

- A a model of random graph homomorphism between two graphs and vary the graphs (gets into computer science questions like graph colorings).
- As a model of random height function/ random surface (analogous to dimers, tilings, SOS, integrable models).

# Perspectives

## History and perspectives

One can view this model from different (not necessarily disjoint) perspectives and flavours.

- A a model of random graph homomorphism between two graphs and vary the graphs (gets into computer science questions like graph colorings).
- As a model of random height function/ random surface (analogous to dimers, tilings, SOS, integrable models).
- Percolation model (level lines / level sets).

## History: random graph homomorphism

- If $G$ is a tree: tree indexed random walk (Benjamini, Peres , 94).
- Introduced by Benjamini, Häggström and Mossel in 2000 studied some properties on general graphs (e.g. tree with leaves wired).
- I. Benjamini and G. Schechtman (maximal height difference)
- Benjamini, Yadin, Yehudayoff : ($n \times n$ torus, range $\geq c\sqrt{\log n}$).
- Ron Peled. In high dimensions , the height function is localized.

## Random surface model: continuous heights

One can consider continuous height functions $\varphi \in \mathbb{R}^{\mathbb{Z}^2}$ with

$$\mathbb{P}(\varphi) \propto \exp(\sum_{u \sim v} U(\phi_u - \phi_v))\delta_0(d\varphi_{\partial\Lambda}) \prod_{v \in V \setminus 0} d\varphi_v$$

We expect this to be delocalized usually

## Random surface model: continuous heights

One can consider continuous height functions $\varphi \in \mathbb{R}^{\mathbb{Z}^2}$ with

$$\mathbb{P}(\varphi) \propto \exp(\sum_{u\sim v} U(\phi_u - \phi_v))\delta_0(d\varphi_{\partial\Lambda}) \prod_{v\in V\setminus 0} d\varphi_v$$

We expect this to be delocalized usually

- $U(x) = x^2$ is the Gaussian free field.
- $U$ twice continuously differentiable (and some further assumptions) on $\mathbb{R}$: Brescamp, Lieb and Lebowitz ('76), and generalized later by Ioffe, Sholshman and Velenik ('02)
- Uniformly convex $U$: Naddaf and Spencer, Miller (scaling limit to GFF), Funaki and Spohn (Gibbs measures for 'tilts'). Techniques include: Brescamp-Lieb inequality, Helffer-Sjostrand representation, homogenization.
- Hammock potential: Peled and Milos (Mermin–Wagner type arguments).

## Random surface model: discrete heights

One can consider discrete height functions $\varphi \in \mathbb{Z}^{\mathbb{Z}^2}$ with

$$\mathbb{P}(\varphi) \propto \exp\Big(\sum_{u \sim v} U(\phi_u - \phi_v)\Big)\delta_0(d\varphi_{\partial\Lambda}) \prod_{v \in V \setminus 0} d\varphi_v$$

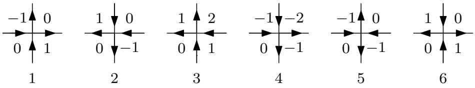
## Random surface model: discrete heights

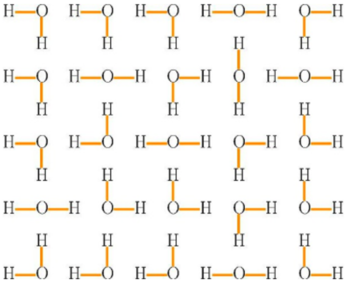
One can consider discrete height functions $\varphi \in \mathbb{Z}^{\mathbb{Z}^2}$ with

$$\mathbb{P}(\varphi) \propto \exp\Big(\sum_{u \sim v} U(\phi_u - \phi_v)\Big)\delta_0(d\varphi_{\partial\Lambda}) \prod_{v \in V \setminus 0} d\varphi_v$$

- Frohlich and Spencer: $U(x) = -\beta|x|$ or $U(x) = -\beta x^2$ . Delocalization for small $\beta$ and localization for large $\beta$ (using a mapping to Coulomb gas). This is called **Roughening transition.**
- Glazman and Manolescu (2019): Delocalization for uniform Lipschitz on triangular lattice (a connection with loop $O(2)$ model is exploited).

# Six vertex / Square ice model



# Six vertex / Square ice model

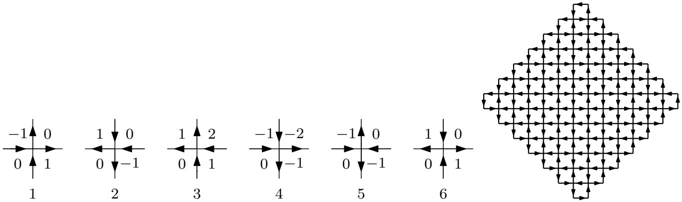



Figure: Put weight $c > 0$ on the last two configurations.

- Our model: $c = 1$. We prove logarithmic variance.

# Six vertex / Square ice model

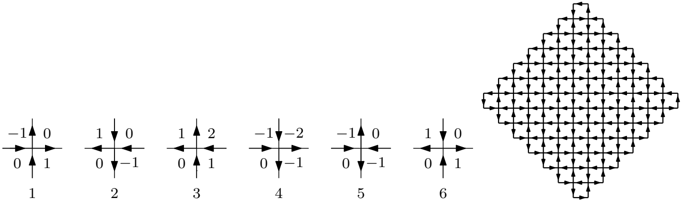



Figure: Put weight $c > 0$ on the last two configurations.

- Our model: $c = 1$. We prove logarithmic variance.
- For $c > 2$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ height function is localized. Recently shown by Duminil-Copin, Harel, Gagnebin, Manolescu, Tassion, '17 (using Bethe Ansatz).
- See Spinka and R' (19) for a short proof for $c > 2$ case.

# Six vertex / Square ice model

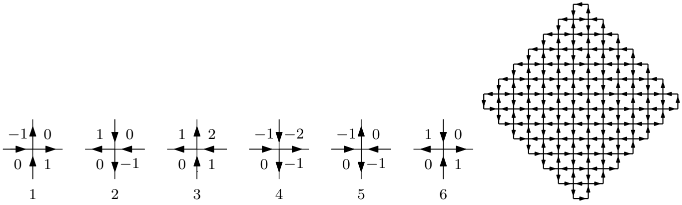



Figure: Put weight $c > 0$ on the last two configurations.

- Our model: $c = 1$. We prove logarithmic variance.
- For $c > 2$ on $\mathbb{Z}_n \times \mathbb{Z}_n$ height function is localized. Recently shown by Duminil-Copin, Harel, Gagnebin, Manolescu, Tassion, '17 (using Bethe Ansatz).
- See Spinka and R' (19) for a short proof for $c > 2$ case.
- Conjecture: If $c \in (0, 2]$ : height function $\to k(c)$Gaussian free field. This is wide open except the **free fermion point** $c = \sqrt{2}$ (dimer model).

## General strategy

Our approach is to adopt renormalization technique for random cluster model developed by Duminil-Copin, Sidorovicius and Tassion to prove the dichotomy theorem.

# Percolation Picture

We will consider the percolation processes induced by $h \in S$.

# Percolation Picture

We will consider the percolation processes induced by $h \in S$.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 2 | | 2 | | 2 | | 2 | | |
| | 2 | 1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 | |
| 0 | 1 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 4 | 3 | 2 | |
| 0 | -1 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 3 | 4 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | -1 | -2 | -1 | 0 | 1 | 2 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 1 | 2 | 3 | 2 | |
| | | 2 | | 2 | | 2 | | 2 | | |

# Percolation Picture

We will consider the percolation processes induced by $h \in S$.

There are two distinct types of connectivity we will need to think about:

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 2 | | 2 | | 2 | | 2 | | |
| | 2 | 1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 | |
| 0 | 1 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 4 | 3 | 2 | |
| 0 | -1 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 3 | 4 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | -1 | -2 | -1 | 0 | 1 | 2 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 1 | 2 | 3 | 2 | |
| | | 2 | | 2 | | 2 | | 2 | | |

# Percolation Picture

We will consider the percolation processes induced by $h \in S$.

There are two distinct types of connectivity we will need to think about:

- the usual connectivity of $\mathbb{Z}^2$, and

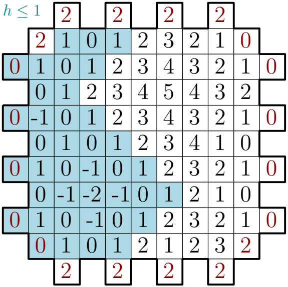
| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | 2 | | 2 | | 2 | | 2 | | |
| | 2 | 1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 | |
| 0 | 1 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 2 | 3 | 4 | 5 | 4 | 3 | 2 | |
| 0 | -1 | 0 | 1 | 2 | 3 | 4 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 3 | 4 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | -1 | -2 | -1 | 0 | 1 | 2 | 1 | 0 | |
| 0 | 1 | 0 | -1 | 0 | 1 | 2 | 3 | 2 | 1 | 0 |
| | 0 | 1 | 0 | 1 | 2 | 1 | 2 | 3 | 2 | |
| | | 2 | | 2 | | 2 | | 2 | | |

# Percolation Picture

We will consider the percolation processes induced by $h \in S$.

There are two distinct types of connectivity we will need to think about:

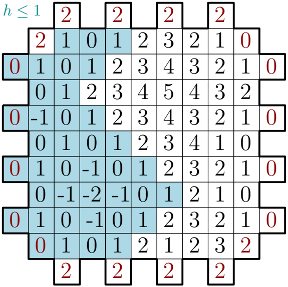
- the usual connectivity of $\mathbb{Z}^2$, and



We will consider the percolation processes induced by $h \in S$.

There are two distinct types of connectivity we will need to think about:

- the usual connectivity of $\mathbb{Z}^2$, and
- $\times$-connectivity, which connects vertices of the same sublattice which are diagonal to one another.

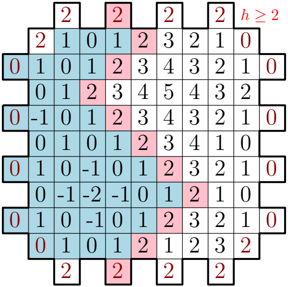


# Percolation Picture

We will consider the percolation processes induced by $h \in S$.

There are two distinct types of connectivity we will need to think about:

- the usual connectivity of $\mathbb{Z}^2$, and
- $\times$-connectivity, which connects vertices of the same sublattice which are diagonal to one another.



# Dichotomy Theorem

We will state our dichotomy theorem for uniform homomorphism model in terms of horizontal $\times$-crossing of rectangles of aspect ratio $\rho$, which we denote $\mathcal{H}^{\times}(\Lambda_{\rho n,n})$:

# Dichotomy Theorem

We will state our dichotomy theorem for uniform homomorphism model in terms of horizontal $\times$-crossing of rectangles of aspect ratio $\rho$, which we denote $\mathcal{H}^{\times}(\Lambda_{\rho n,n})$:

**Theorem (DCHLRR, 19+)**

For the uniform homomorphism, either:

# Dichotomy Theorem

We will state our dichotomy theorem for uniform homomorphism model in terms of horizontal $\times$-crossing of rectangles of aspect ratio $\rho$, which we denote $\mathcal{H}^{\times}(\Lambda_{\rho n,n})$:

**Theorem (DCHLRR, 19+)**

For the uniform homomorphism, either:

- $\phi^{0}_{\Lambda_n}[h_0 > r] < e^{-kr^{\alpha}}$, for some $k, \alpha > 0$, or

# Dichotomy Theorem

We will state our dichotomy theorem for uniform homomorphism model in terms of horizontal $\times$-crossing of rectangles of aspect ratio $\rho$, which we denote $\mathcal{H}^{\times}(\Lambda_{\rho n,n})$:

**Theorem (DCHLRR, 19+)**

For the uniform homomorphism, either:

- $\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr^{\alpha}}$, for some $k, \alpha > 0$, or
- there exists $c(k, r, \rho)$ such that, for any $r$, $k > (2 + \rho)$, and $n$ large enough,

$$c < \phi^0_{\Lambda_{kn}}[\mathcal{H}^{\times}_{h=r}(\Lambda_{\rho n,n})] < 1 - c.$$

# Dichotomy Theorem

We will state our dichotomy theorem for uniform homomorphism model in terms of horizontal $\times$-crossing of rectangles of aspect ratio $\rho$, which we denote $\mathcal{H}^{\times}(\Lambda_{\rho n,n})$:

**Theorem (DCHLRR, 19+)**

For the uniform homomorphism, either:

- ~~$\phi^0_{\Lambda_n}[h_0 > r] < e^{-kr^{\alpha}}$, for some $k, \alpha > 0$~~[Chandgotia, Peled, Sheffield, Tassy ['19]]
- there exists $c(k, r, \rho)$ such that, for any $r$, $k > (2 + \rho)$, and $n$ large enough,

$$c < \phi^0_{\Lambda_{kn}}[\mathcal{H}^{\times}_{h=r}(\Lambda_{\rho n,n})] < 1 - c.$$

# Dichotomy Theorem

There are two distinct elements required for the dichotomy argument:

# Dichotomy Theorem

There are two distinct elements required for the dichotomy argument:

- A relation between horizontal and vertical crossings,

# Dichotomy Theorem

There are two distinct elements required for the dichotomy argument:

- A relation between horizontal and vertical crossings,

$$\phi^0_{\mathbb{S}_n}[\mathcal{H}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})] \geq c\left(\phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})]\right)^{\rho/c},$$

  where $\mathbb{S}_n$ is the infinite strip of height $2n$.

# Dichotomy Theorem

There are two distinct elements required for the dichotomy argument:

- A relation between horizontal and vertical crossings,

$$\phi^0_{\mathbb{S}_n}[\mathcal{H}^\times_{h\geq 2}(\Lambda_{\rho n,n})] \geq c\left(\phi^0_{\mathbb{S}_n}[\mathcal{V}^\times_{h\geq 2}(\Lambda_{\rho n,n})]\right)^{\rho/c},$$

  where $\mathbb{S}_n$ is the infinite strip of height $2n$.

- A renormalization argument, which will use the generalized RSW estimate above to prove that

$$\phi^0_{\Lambda_{20n}}[\exists \times\text{-circuit of } h \geq 2 \text{ in } \Lambda_{20n} \setminus \Lambda_{10n}] \leq C \cdot \phi^0_{\Lambda_{2n}}[\exists \times\text{-circuit of } h \geq 2 \text{ in } \Lambda_{2n} \setminus \Lambda_n]^2.$$

# Tools for the proof

The uniform homomorphism has a few good properties:

# Tools for the proof

The uniform homomorphism has a few good properties:

- $h$ satisfies the FKG inequality — that is,

$$\phi^0_R[A \cap B] \geq \phi^0_R[A] \cdot \phi^0_R[B], \text{ for any } A \text{ and } B \text{ increasing in } h.$$

# Tools for the proof

The uniform homomorphism has a few good properties:

- *h* satisfies the FKG inequality — that is,

$$\phi_R^0[A \cap B] \geq \phi_R^0[A] \cdot \phi_R^0[B], \text{ for any } A \text{ and } B \text{ increasing in } h.$$

- *h* has the $\times$-Domain Markov Property.

# Tools for the proof

The uniform homomorphism has a few good properties:

- $h$ satisfies the FKG inequality — that is,

$$\phi_R^0[A \cap B] \geq \phi_R^0[A] \cdot \phi_R^0[B], \text{ for any } A \text{ and } B \text{ increasing in } h.$$

- $h$ has the $\times$-Domain Markov Property.
- Under 'good' boundary conditions, there are several equivalent ways to express crossing events:

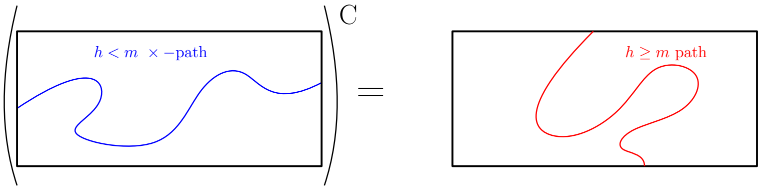
# The ‘free lunch’ equalities

The planar duality of the crossings implies that

# The ‘free lunch’ equalities

The planar duality of the crossings implies that

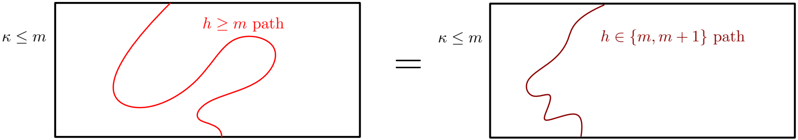
$$\mathcal{H}^{\times}_{h<m}(R)^c = \mathcal{V}_{h\geq m}(R)$$



# The 'free lunch' equalities

Suppose the boundary conditions on the horizontal sides of $R$ are below $m$. Then

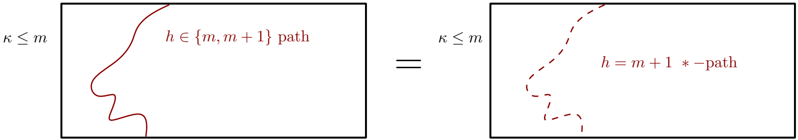
$$\mathcal{H}^{\times}_{h<m}(R)^c = \mathcal{V}_{h\geq m}(R) = \mathcal{V}_{h\in\{m,m+1\}}(R)$$



# The ‘free lunch’ equalities

Suppose the boundary conditions on the horizontal sides of $R$ are below $m$. Then

$$\mathcal{H}^{\times}_{h<m}(R)^c = \mathcal{V}_{h\geq m}(R) = \mathcal{V}_{h\in\{m,m+1\}}(R) = \mathcal{V}^{*}_{h=m+1}(R)$$




where $*$-paths connect vertices at $\ell^1$-distance 2.

# The less-than-advantageous properties

There are also some major difficulties in the analysis:

# The less-than-advantageous properties

There are also some major difficulties in the analysis:

- The crossing events $\mathcal{H}^\times$ and $\mathcal{V}^\times$ are not self-dual.

# The less-than-advantageous properties

There are also some major difficulties in the analysis:

- The crossing events $\mathcal{H}^{\times}$ and $\mathcal{V}^{\times}$ are not self-dual.
- The spin space of $h$ is unbounded in both directions.

# The less-than-advantageous properties

There are also some major difficulties in the analysis:

- The crossing events $\mathcal{H}^\times$ and $\mathcal{V}^\times$ are not self-dual.
- The spin space of *h* is unbounded in both directions.

This makes it tricky to 'push' boundary conditions to manipulate the geometry of domains, as there are no optimal boundary conditions for increasing events.

# The less-than-advantageous properties

There are also some major difficulties in the analysis:

- The crossing events $\mathcal{H}^\times$ and $\mathcal{V}^\times$ are not self-dual.
- The spin space of $h$ is unbounded in both directions.

This makes it tricky to ‘push’ boundary conditions to manipulate the geometry of domains, as there are no optimal boundary conditions for increasing events.

To get around this difficulty, we will work with the absolute value of $h$ — which, it turns out, is FKG!

# The less-than-advantageous properties

There are also some major difficulties in the analysis:

- The crossing events $\mathcal{H}^\times$ and $\mathcal{V}^\times$ are not self-dual.
- The spin space of $h$ is unbounded in both directions.

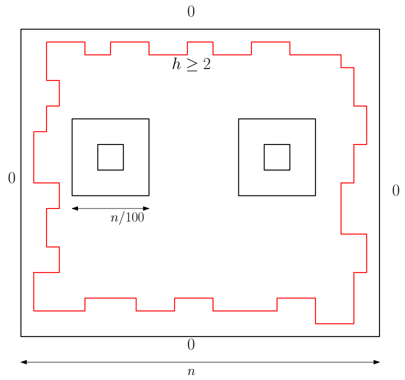
This makes it tricky to 'push' boundary conditions to manipulate the geometry of domains, as there are no optimal boundary conditions for increasing events.

To get around this difficulty, we will work with the absolute value of $h$ — which, it turns out, is FKG! (for good boundary conditions)

# Renormalization

## Step 1: Setup

Let $a_n$ be the probability of a loop with values $\geq 2$ (red loop).

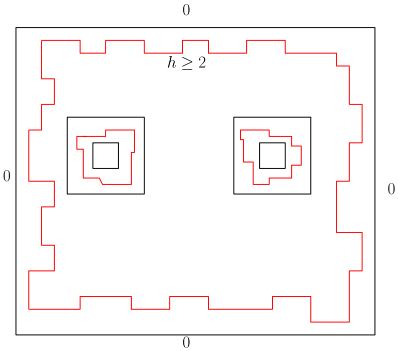



Goal: To show there exists $c > 0$ such that for all $n$, $a_n \geq c$

# Renormalization

## Step 2: Easy Russo Seymour Welsh

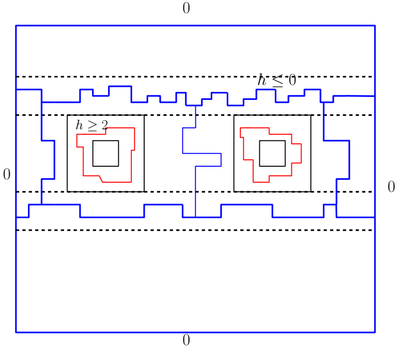
Conditionally on the outermost loop, we can find two inner loops of $h \geq 2$ with positive probability.



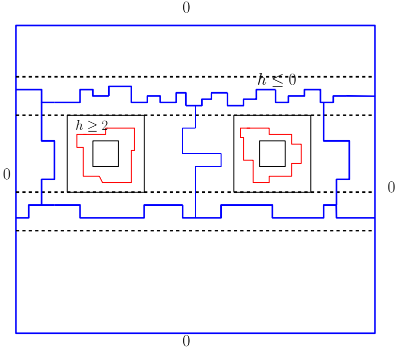
# Renormalization

## Step 3: Hard Russo Seymour Welsh

Forget the outer red loops (the inequality works in our direction). Conditionally on both the inner red loops, we can find two (blue) loops of $h \leq 0$ with positive probability. This is an application of the RSW step and FKG.



# Renormalization




This decouples the red loops. We obtain (after some work) $\exists C, c > 0$ such that $\forall n \geq 1$,

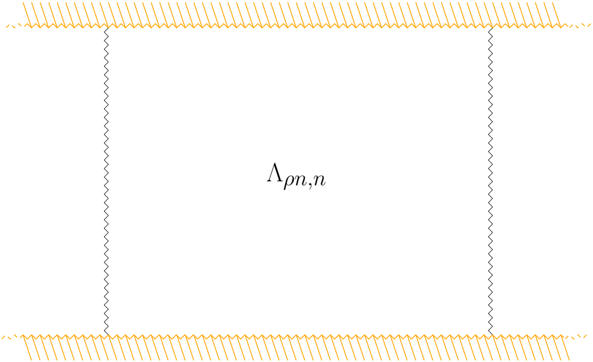
$$a_n \leq C a_{n/100}^2 \implies \text{ either } a_n \geq c \text{ or } a_n \leq C e^{-cn^\alpha}.$$

# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.

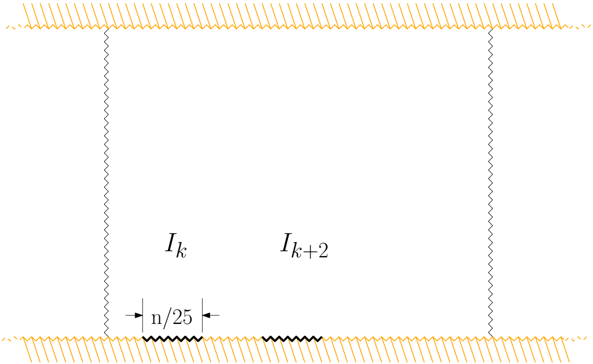
# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.



# Russo-Seymour-Welsh Theory

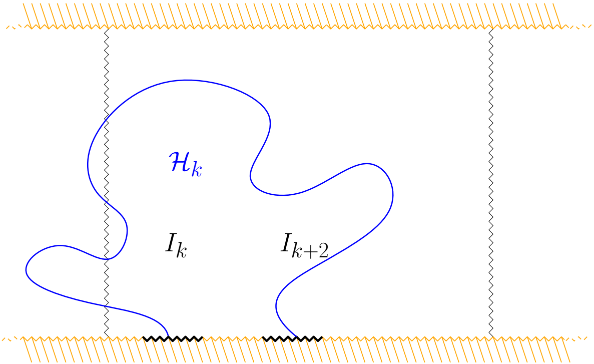
Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.



# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.

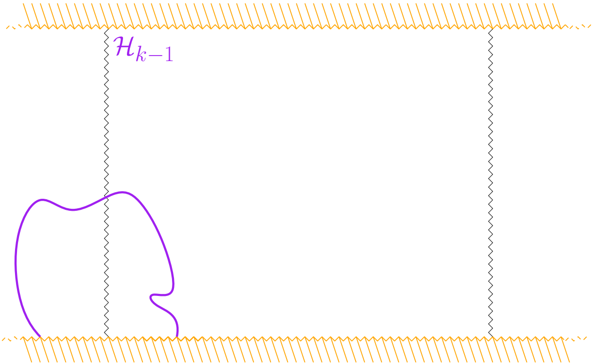
Let $\mathcal{H}_k$ be the event that $I_k$ and $I_{k+2}$ are connected by a $\times$-path of $h \geq 2$.



# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.

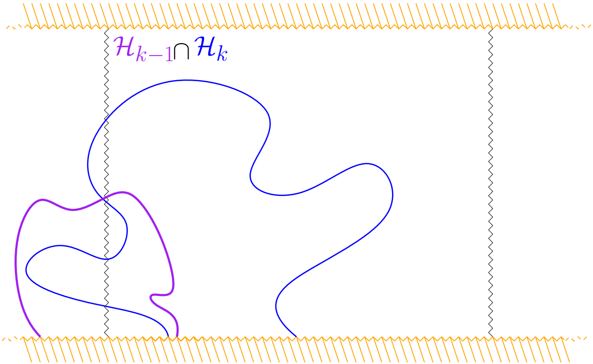
Let $\mathcal{H}_k$ be the event that $I_k$ and $I_{k+2}$ are connected by a $\times$-path of $h \geq 2$.



# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.

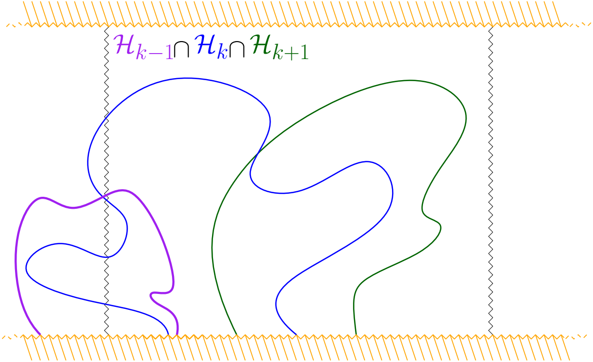
Let $\mathcal{H}_k$ be the event that $I_k$ and $I_{k+2}$ are connected by a $\times$-path of $h \geq 2$.



# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.

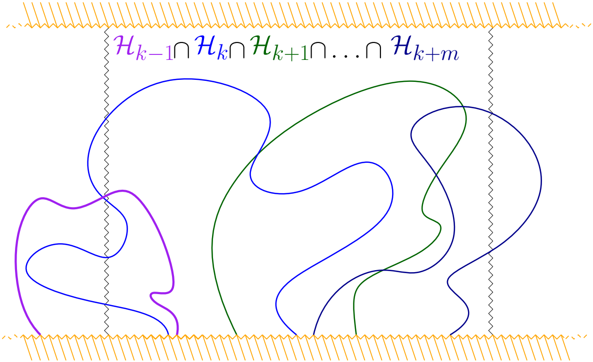
Let $\mathcal{H}_k$ be the event that $I_k$ and $I_{k+2}$ are connected by a $\times$-path of $h \geq 2$.



# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.

Let $\mathcal{H}_k$ be the event that $I_k$ and $I_{k+2}$ are connected by a $\times$-path of $h \geq 2$.

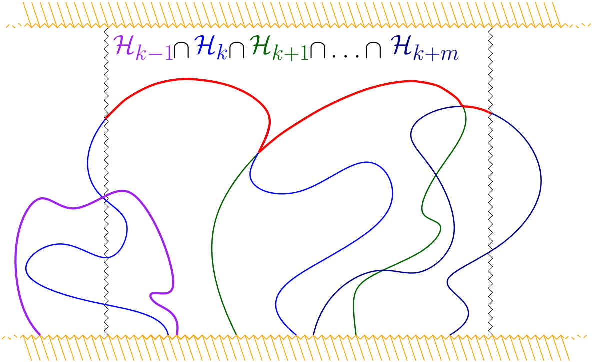


# Russo-Seymour-Welsh Theory

Consider the strip $\mathbb{S}_n$, the rectangle $\Lambda_{\rho n,n}$, and the segments $\{I_k\}$.

Let $\mathcal{H}_k$ be the event that $I_k$ and $I_{k+2}$ are connected by a $\times$-path of $h \geq 2$.

The intersection of (at most) $(25\rho + 1)$ $\mathcal{H}_i$'s implies the existence of a horizontal crossing of $\Lambda_{\rho n,n}$.

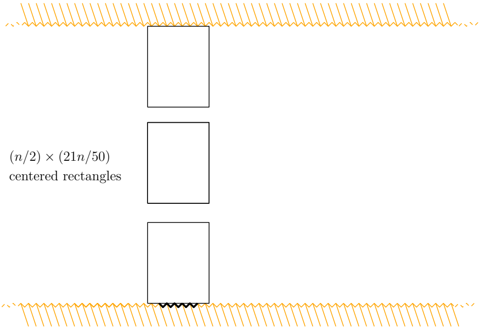


# Russo-Seymour-Welsh Theory

By a union bound, the probability of connecting any particular $I_k$ to the top is comparable to $\phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})]$.

# Russo-Seymour-Welsh Theory

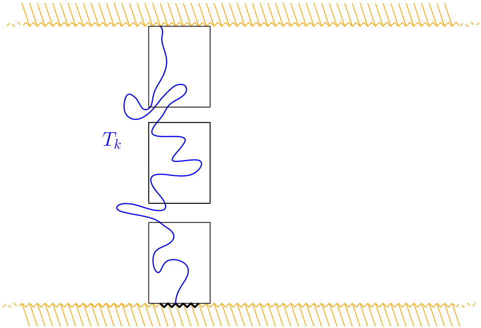
By a union bound, the probability of connecting any particular $I_k$ to the top is comparable to $\phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})]$.



# Russo-Seymour-Welsh Theory

By a union bound, the probability of connecting any particular $I_k$ to the top is comparable to $\phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})]$.

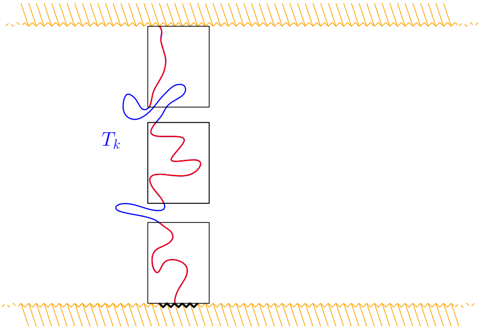
We define $T_k$ to be the event in the picture, which restricts the geometry of the crossing path.



# Russo-Seymour-Welsh Theory

By a union bound, the probability of connecting any particular $I_k$ to the top is comparable to $\phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})]$.

We define $T_k$ to be the event in the picture, which restricts the geometry of the crossing path.

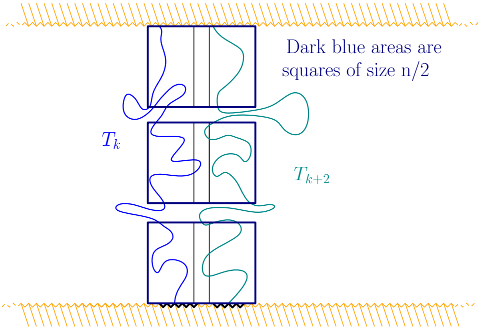


# Russo-Seymour-Welsh Theory

By a union bound, the probability of connecting any particular $I_k$ to the top is comparable to $\phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})]$.

We define $T_k$ to be the event in the picture, which restricts the geometry of the crossing path.

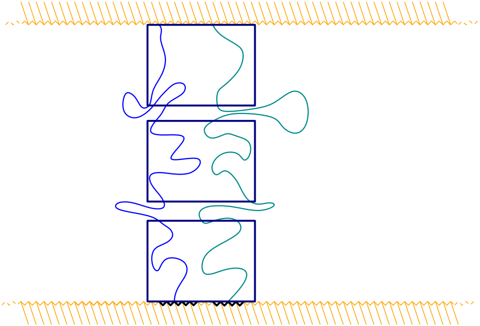
When $T_k$ and $T_{k+2}$ occur simultaneously, we have three squares that are doubly crossed by $\times$-paths of $h \geq 2$.



# Russo-Seymour-Welsh Theory

We now make a (rather major) assumption:

$$\phi^0_{\mathbb{S}_n}[T_k] > c(\rho) \cdot \phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})].$$

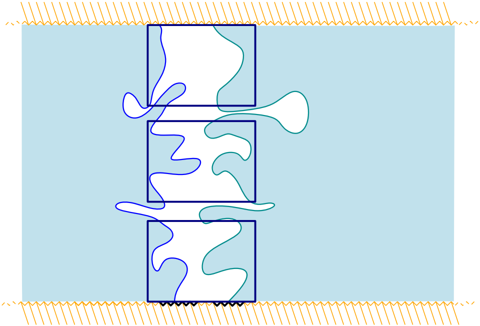


# Russo-Seymour-Welsh Theory

We now make a (rather major) assumption:

$$\phi^0_{\mathbb{S}_n}[T_k] > c(\rho) \cdot \phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})].$$

Condition on the value of $h$ to the left of the leftmost path satisfying $T_k$, and to the right of the rightmost path satisfying $T_{k+2}$.

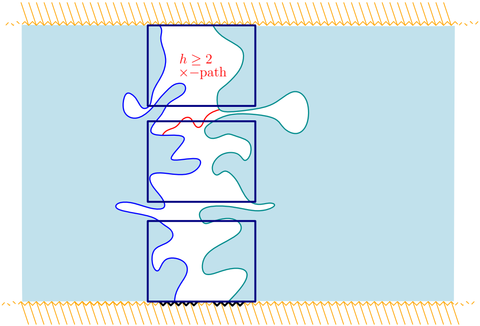


# Russo-Seymour-Welsh Theory

We now make a (rather major) assumption:

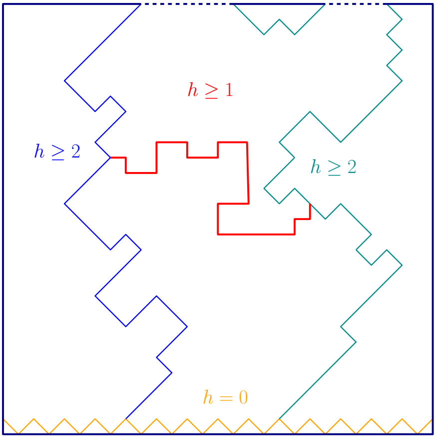
$$\phi^0_{\mathbb{S}_n}[T_k] > c(\rho) \cdot \phi^0_{\mathbb{S}_n}[\mathcal{V}^{\times}_{h\geq 2}(\Lambda_{\rho n,n})].$$

It will be sufficient to prove that probability of crossing the white region horizontally is bounded below by a constant.



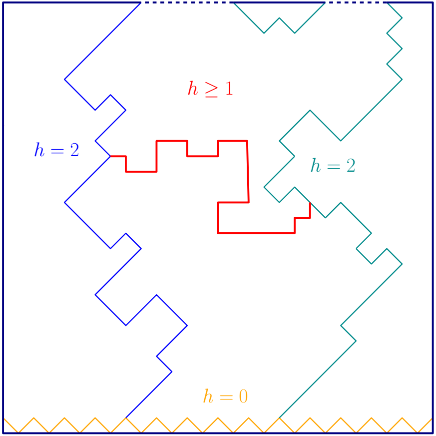
# RSW Proof: Step 1

We zoom in on the bottom square $S^{-}$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



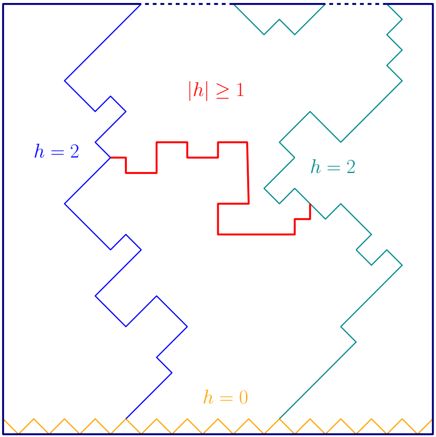
# RSW Proof: Step 1

We zoom in on the bottom square $S^{-}$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .

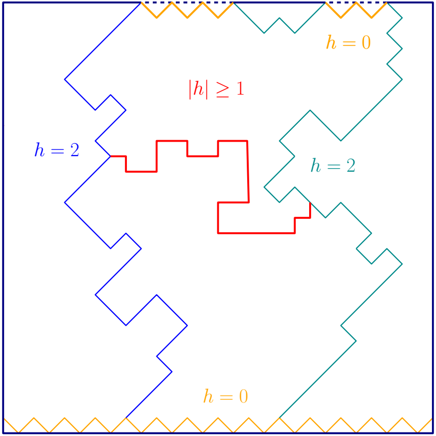


# RSW Proof: Step 1

We zoom in on the bottom square $S^{-}$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .

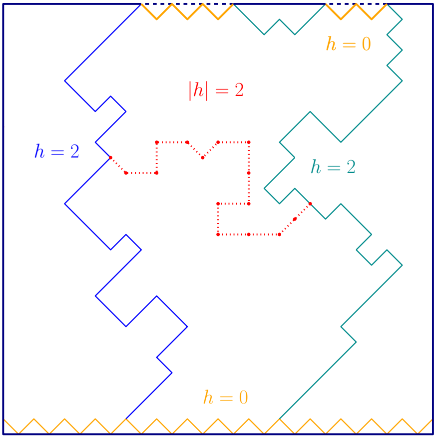


We zoom in on the bottom square $S^{-}$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



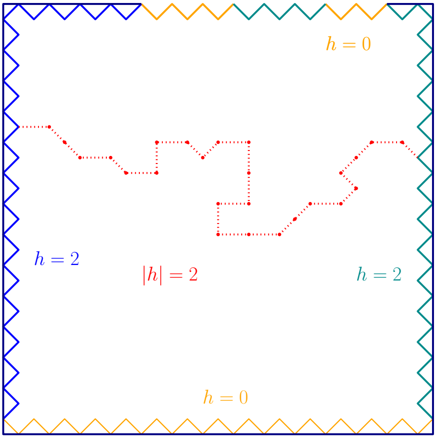
# RSW Proof: Step 1

We zoom in on the bottom square $S^{-}$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



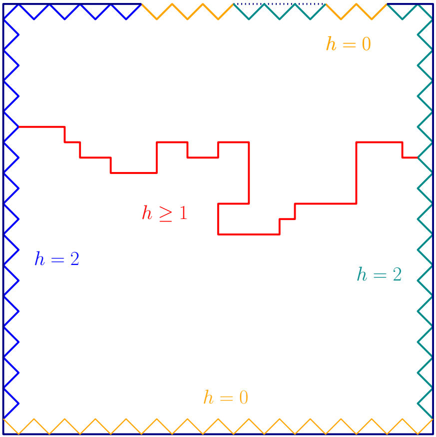
# RSW Proof: Step 1

We zoom in on the bottom square $S^{-}$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



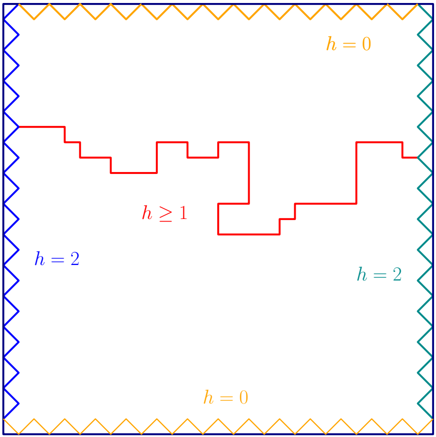
# RSW Proof: Step 1

We zoom in on the bottom square $S^-$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



# RSW Proof: Step 1

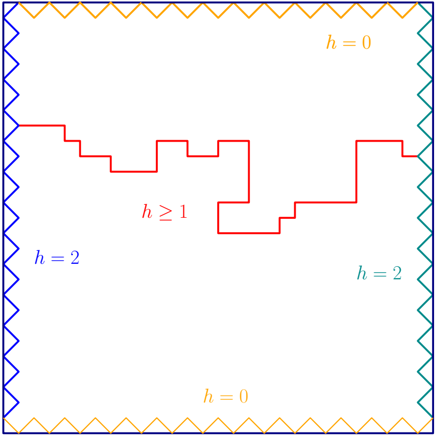
We zoom in on the bottom square $S^{-}$, and consider the event $\bar{H}$, where the right boundary is connected to the left by $h \geq 1$ path. .



# RSW Proof: Step 1

Thus, we deduce that the probability of $\bar{H}$ is bounded below by

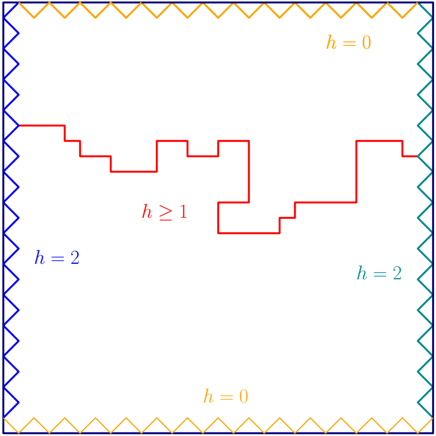
$$\phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)] \qquad .$$



# RSW Proof: Step 1

Thus, we deduce that the probability of $\bar{H}$ is bounded below by

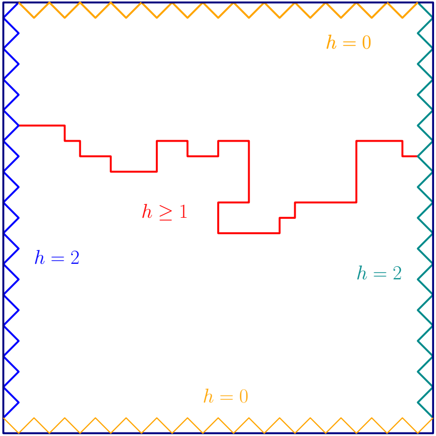
$$\phi^{0/2}_{S^-}[\mathcal{H}_{h\geq 1}(S^-)] = 1 - \phi^{0/2}_{S^-}[\mathcal{V}^{\times}_{h\leq 0}(S^-)] \qquad .$$



# RSW Proof: Step 1

Thus, we deduce that the probability of $\bar{H}$ is bounded below by

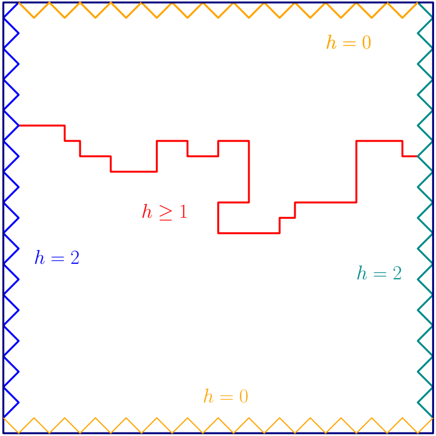
$$\phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)] = 1 - \phi_{S^-}^{0/2}[\mathcal{V}_{h\leq 0}^{\times}(S^-)] \geq 1 - \phi_{S^-}^{0/2}[\mathcal{V}_{h\leq 1}(S^-)].$$



# RSW Proof: Step 1

Thus, we deduce that the probability of $\bar{H}$ is bounded below by

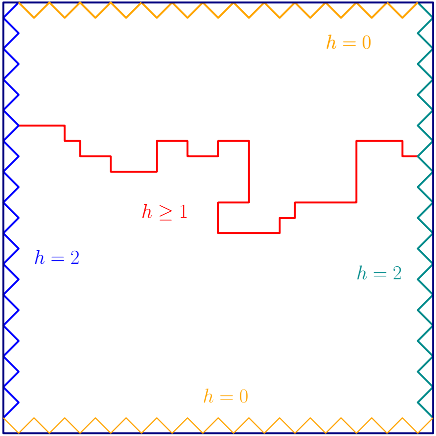
$$\phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)] = 1 - \phi_{S^-}^{0/2}[\mathcal{V}_{h\leq 0}^{\times}(S^-)] \geq 1 - \phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)].$$



# RSW Proof: Step 1

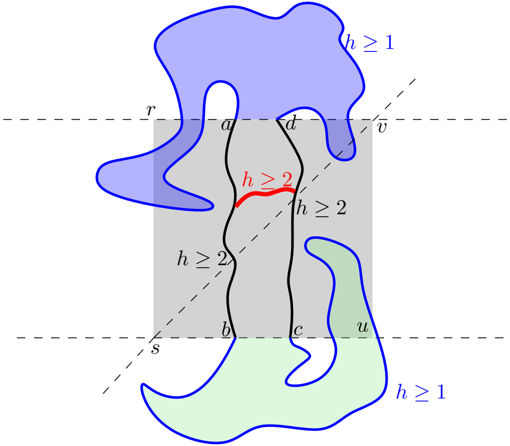
Thus, we deduce that the probability of $\bar{H}$ is bounded below by

$$\phi_{S^-}^{0/2}[\mathcal{H}_{h\geq 1}(S^-)] \geq 1/2$$



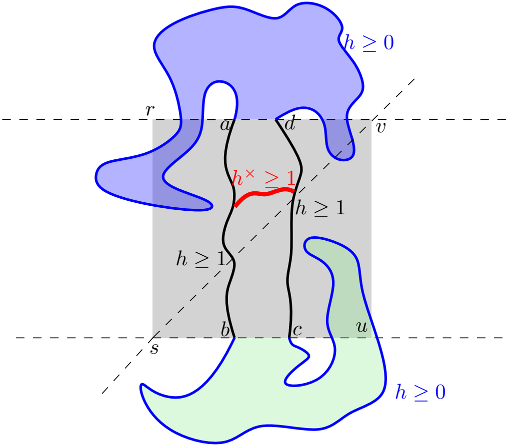
# RSW Proof: Step 2

We zoom in on the middle square $S$, and look for a $h \geq 2$ $\times$-crossing.



# RSW Proof: Step 2

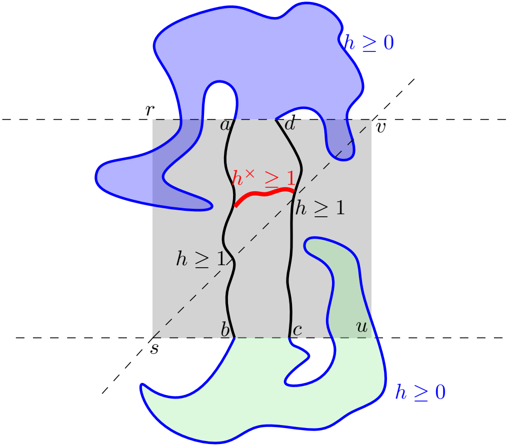
We zoom in on the middle square $S$, and look for a $h \geq 1$ $\times$-crossing.



# RSW Proof: Step 2

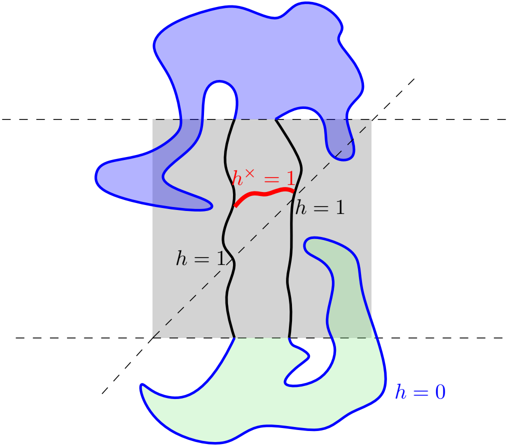
We zoom in on the middle square $S$, and look for a $h \geq 1$ $\times$-crossing.

Unlike before, we cannot push boundary conditions of $h = 0$ in, because $h \geq 1$ is *not* the same as $|h| \geq 1$!



# RSW Proof: Step 2

We look for a symmetric domain in other ways:



# RSW Proof: Step 2

We look for a symmetric domain in other ways:

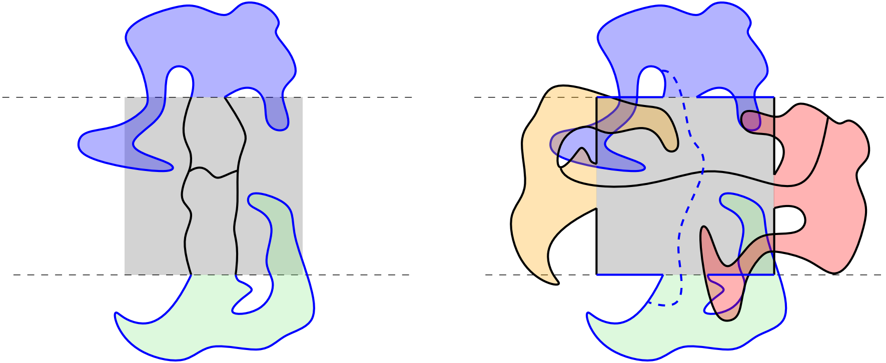
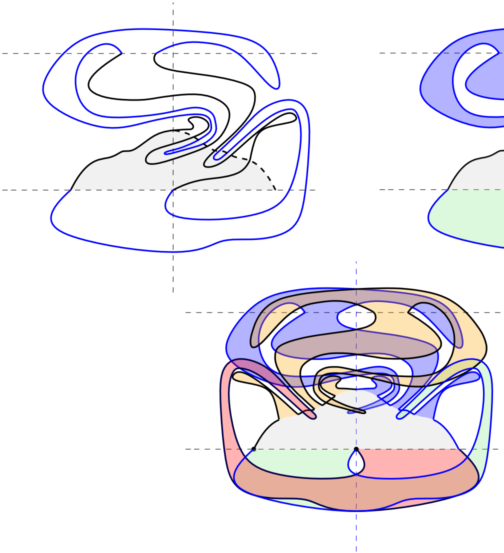


Figure: Blue is $h = 0$, $\times$ and black is $h = 1$, $\times$

# RSW Proof: Step 2

We look for a symmetric domain in other ways:



Thank you!