A short proof of the discontinuity of phase transition in the planar random-cluster model with $q > 4$

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Joint with Gourab Ray

University of British Columbia

Bristol, July, 2020

Percolation

Percolation on $\mathbb{Z}^2$:

- Parameter: $p \in [0,1]$
- Independently for each edge:
  open (keep) it with probability p,
  close (delete) it with probability $1-p$.
- Random subgraph: $\omega \in \{0,1\}^{E(\mathbb{Z}^2)}$

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Does ω have an **infinite open cluster**?

1. **No** if $p \leq \frac{1}{2}$ [Harris 60]
2. **Yes** if $p > \frac{1}{2}$ [Kesten 80]

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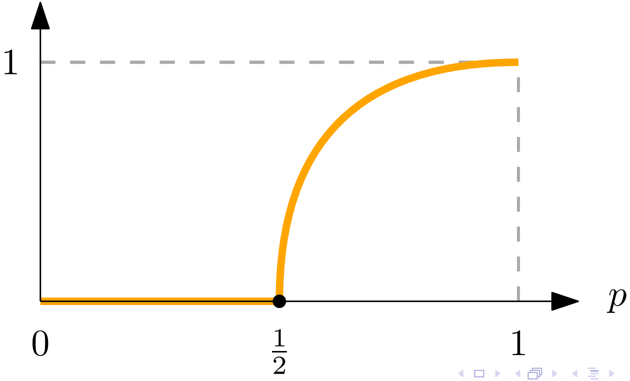
$$\theta(p) := \mathbb{P}_p(\text{the origin is in an } \textbf{infinite open cluster} \text{ of } \omega)$$



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What happens at p_c?

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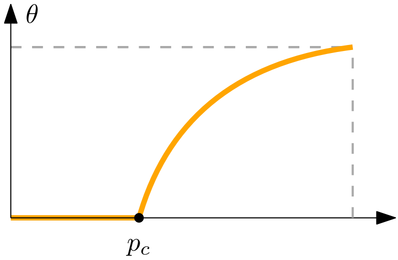

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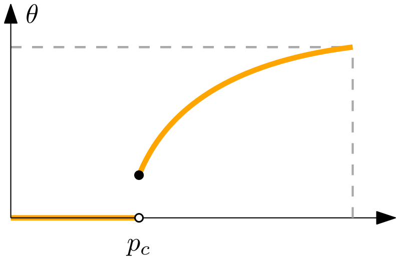

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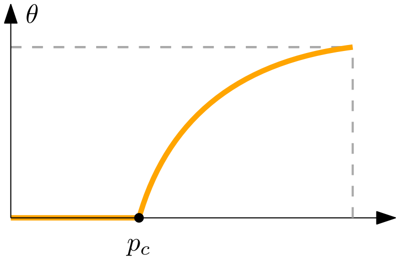

- $d = 2$ [Harris, Kesten]
- $d \geq 19$ [Hara–Slade 94]

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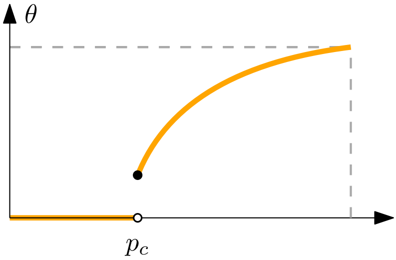

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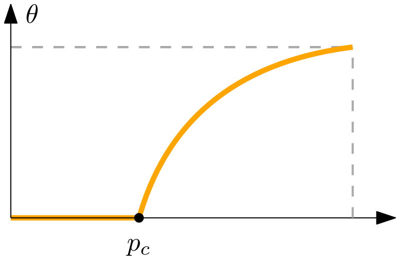

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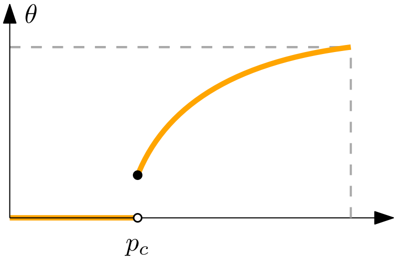

- Not expected for any d
- Open problem!

The random-cluster model Definition

- Finite graph $G = (V, E)$
- Two parameters: $p \in [0, 1]$ and $q > 0$
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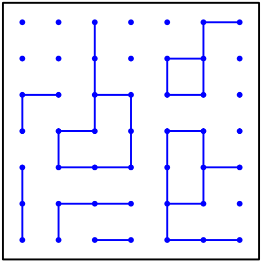

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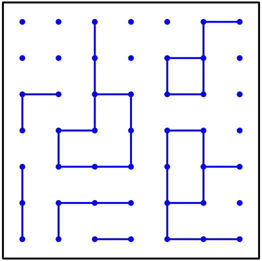

$\#\text{clusters} = 7 + 11 = 18$

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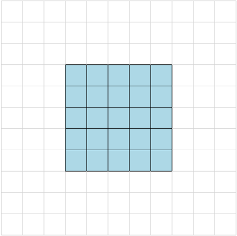

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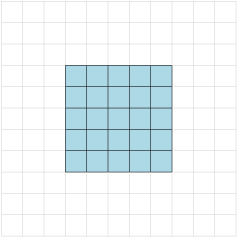

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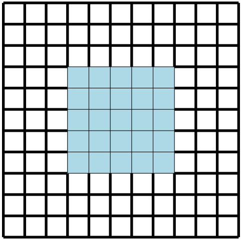
2. The **wired** measure



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$$\theta^{\text{free}}(p) := \mathbb{P}^{\text{free}}_{p,q}(\text{the origin is in an infinite open cluster of } \omega)$$
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1. $\theta^{\text{wired}}(p) = \theta^{\text{free}}(p) = 0$ if $p < p_c$
2. $\theta^{\text{wired}}(p) \geq \theta^{\text{free}}(p) > 0$ if $p > p_c$

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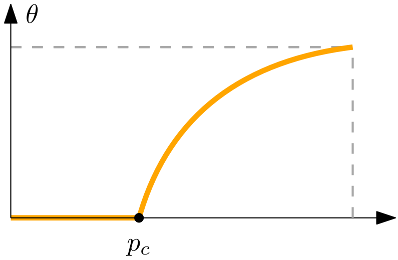

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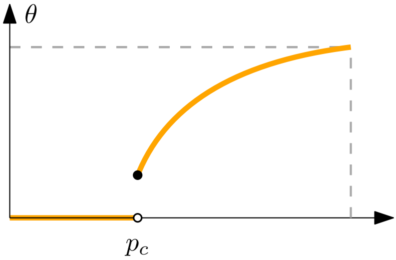

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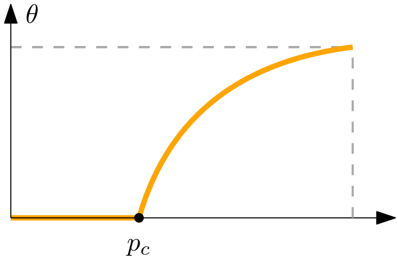

- $\theta^{\text{wired}}(p_c) = 0$

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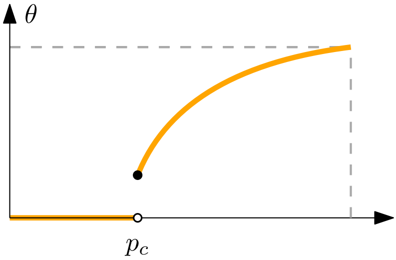

- $\theta^{\text{wired}}(p_c) > 0$

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   - Short proof [Ray–Spinka 2019]

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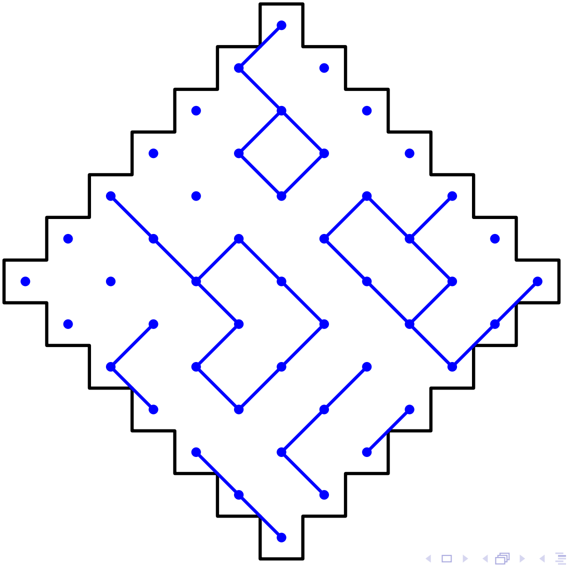
2. Height function representation for the six-vertex model

From the random-cluster model to loop configurations



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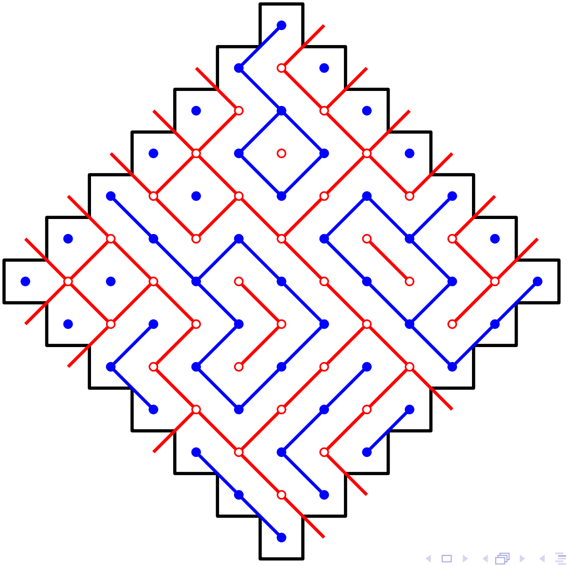

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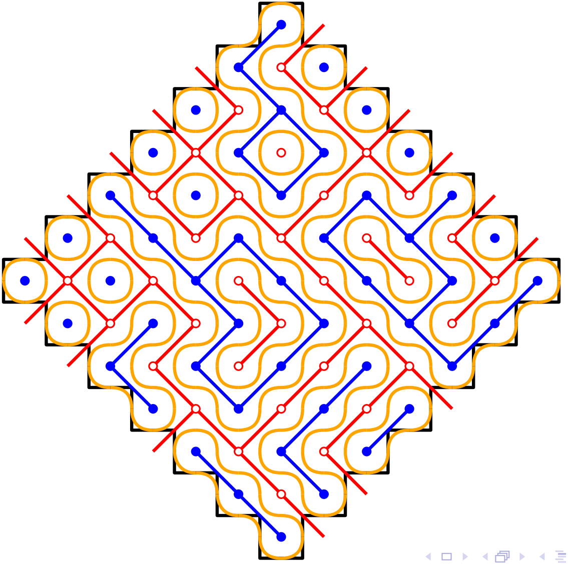

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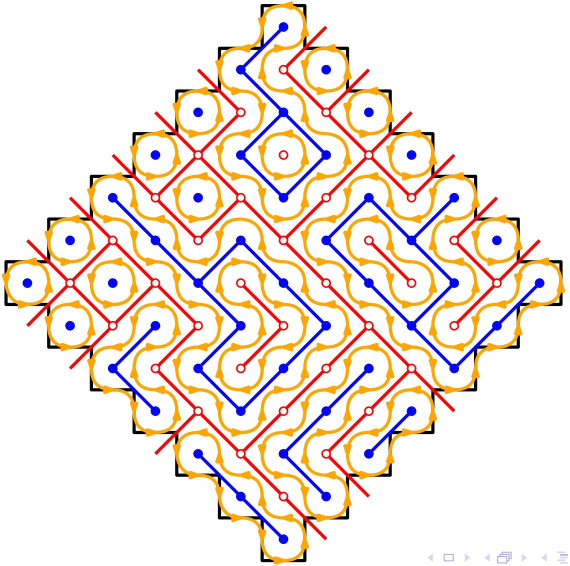

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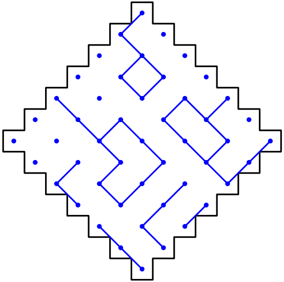

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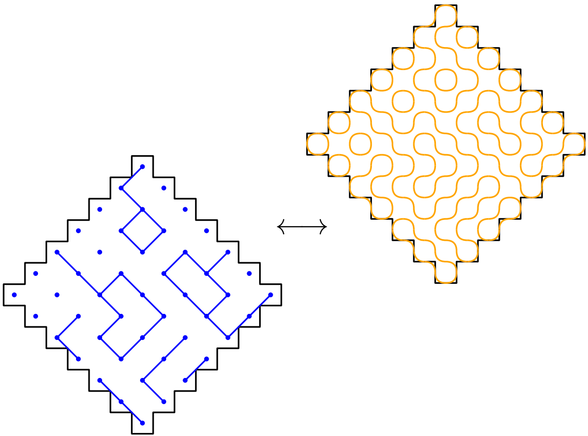

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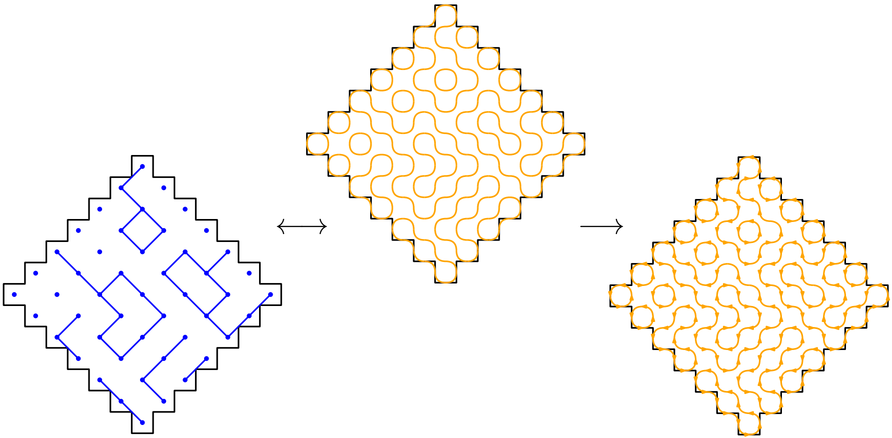

The six-vertex model

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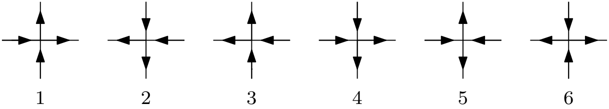

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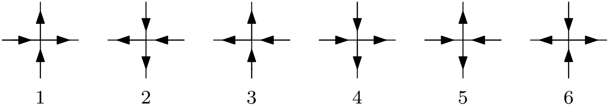

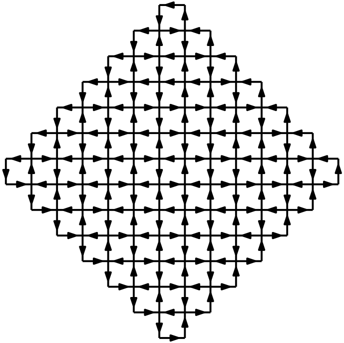

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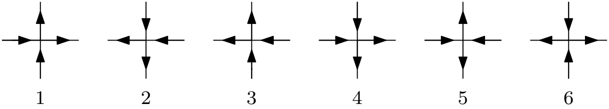

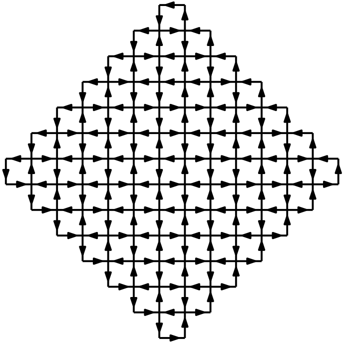

$$\mathbb{P}_c(\sigma) \propto c^{\#\{\text{type 5 and 6 vertices}\}}$$

From the six-vertex model to loop configurations

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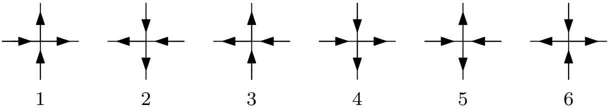

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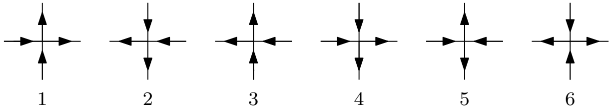

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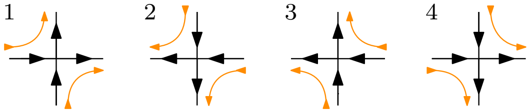

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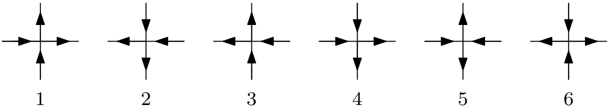

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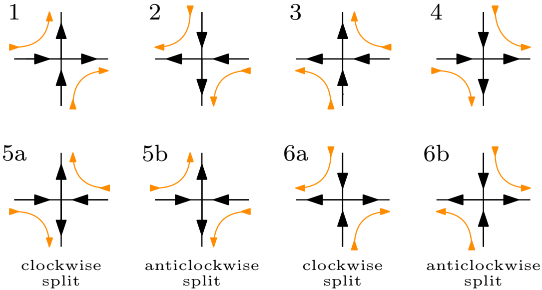

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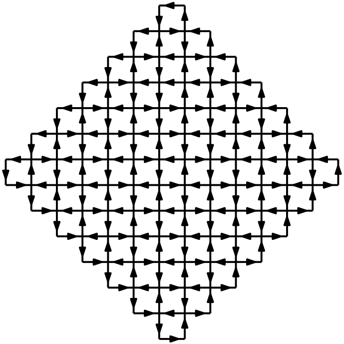

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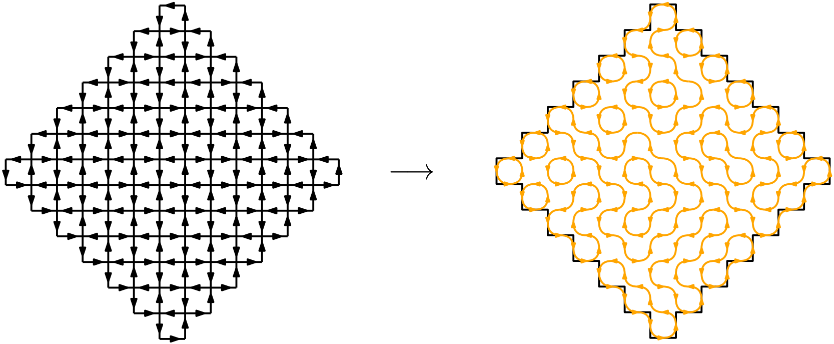
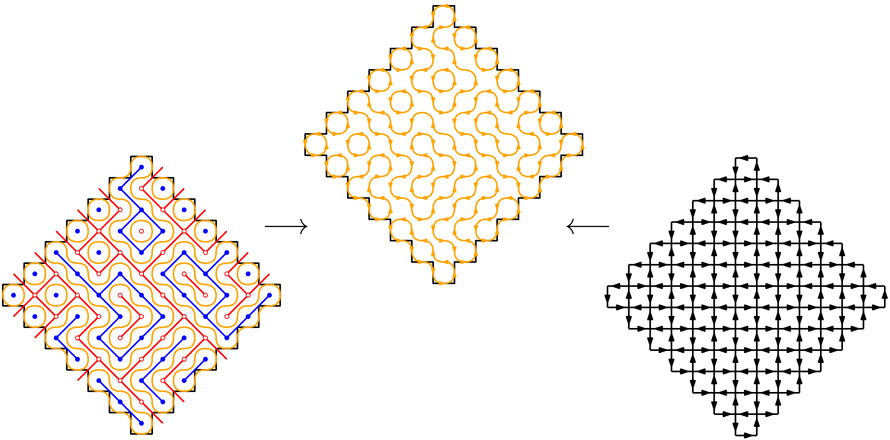


The BKW coupling



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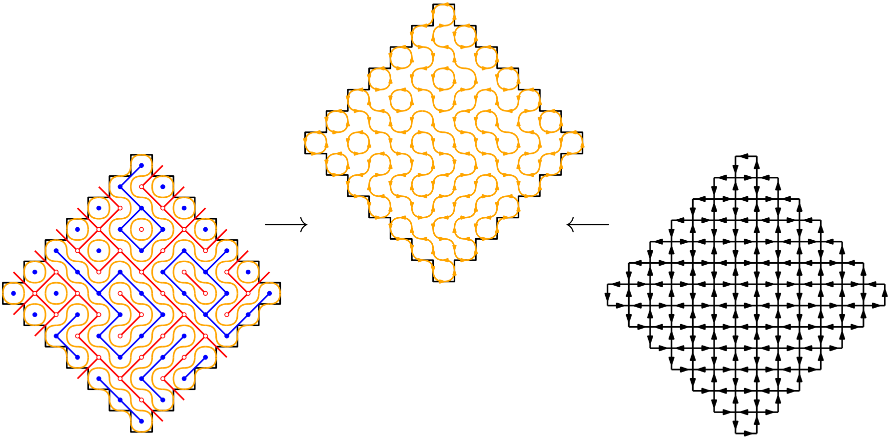

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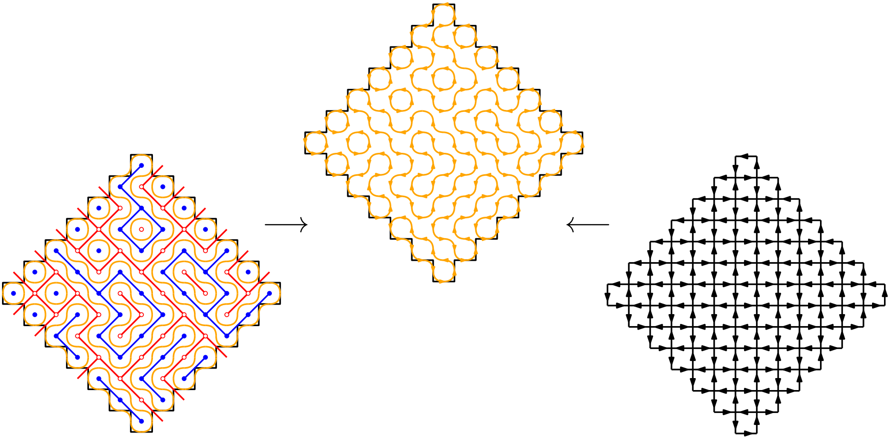

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$p = p_c = \frac{\sqrt{q}}{1+\sqrt{q}}$ and **Euler's formula**

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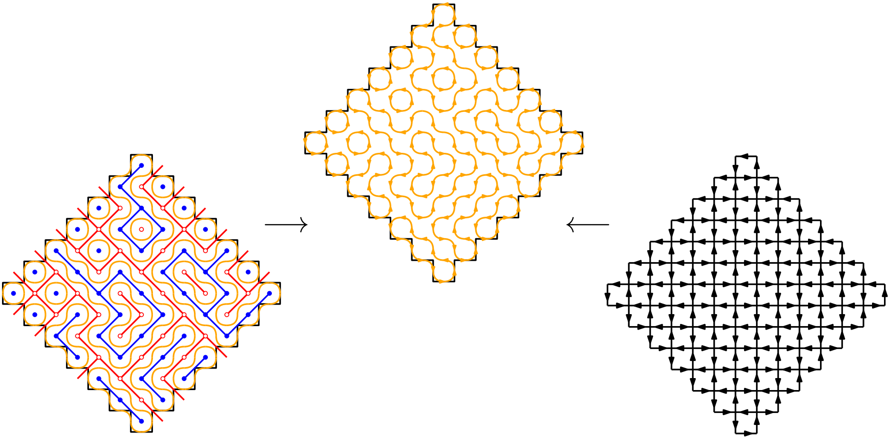
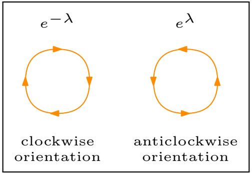
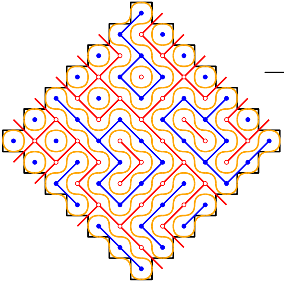


$\sqrt{q}^{\#\text{loops}}$ $c^{\#\{\text{type 5 and 6}\}}$

The BKW coupling





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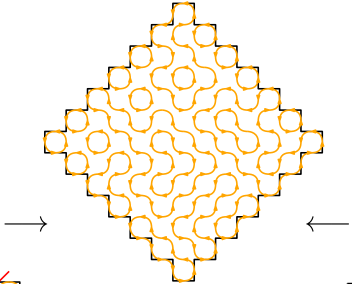
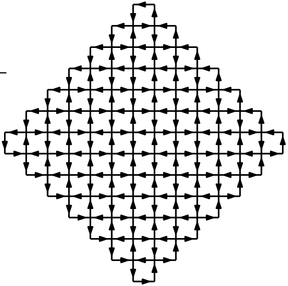
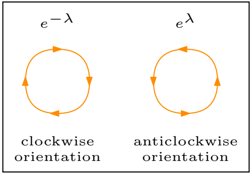


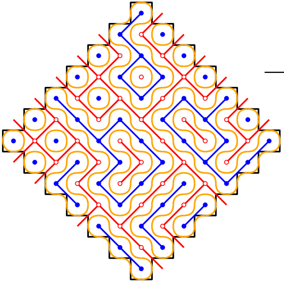


$c^{\#\{\text{type 5 and 6}\}}$

$$\sqrt{q} = e^{-\lambda} + e^{\lambda}$$

The BKW coupling





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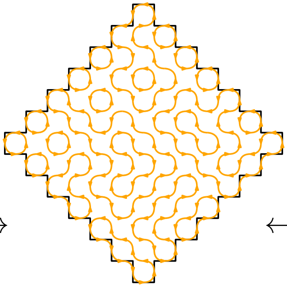
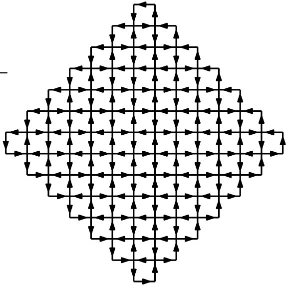
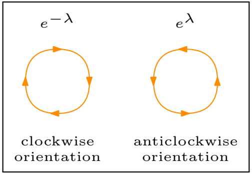


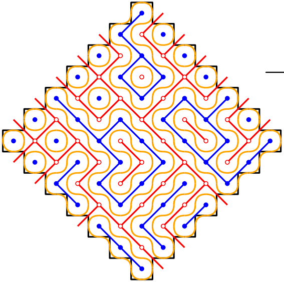
$e^{\lambda \# \{\text{signed loops}\}}$



$c^{\# \{\text{type 5 and 6}\}}$

The BKW coupling





$\sqrt{q}^{\#\text{loops}}$

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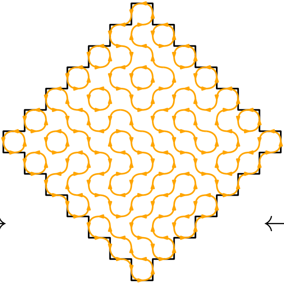
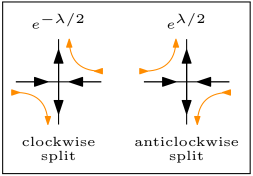
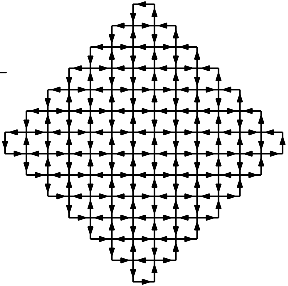


$e^{\lambda \#\{\text{signed loops}\}}$

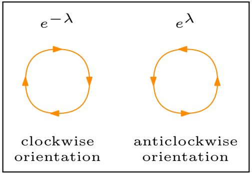


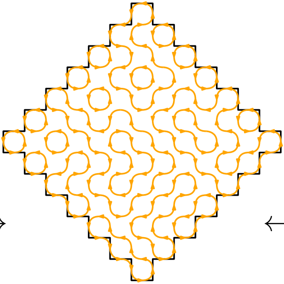


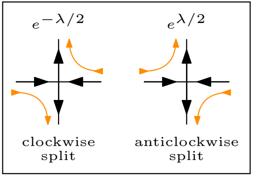
$c^{\#\{\text{type 5 and 6}\}}$

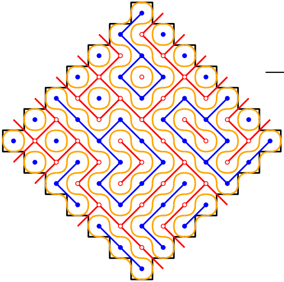
$$c = e^{-\lambda/2} + e^{\lambda/2}$$

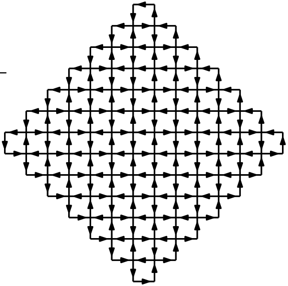
The BKW coupling











$\sqrt{q}^{\#\text{loops}}$

$$\sqrt{q} = e^{-\lambda} + e^{\lambda}$$

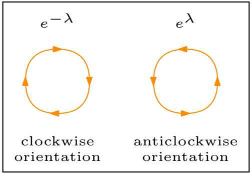
$e^{\lambda \#\{\text{signed loops}\}}$

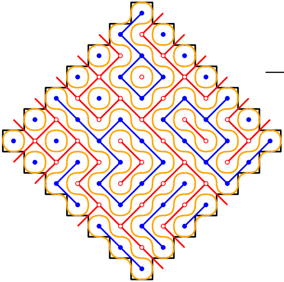
$e^{\frac{\lambda}{2}\#\{\text{signed splits}\}}$

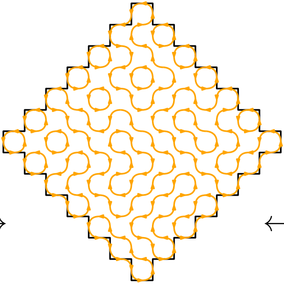
$c^{\#\{\text{type 5 and 6}\}}$

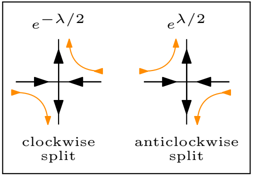
$$c = e^{-\lambda/2} + e^{\lambda/2}$$

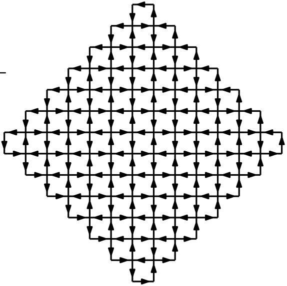
The BKW coupling











$\sqrt{q}^{\#\text{loops}}$

$$\sqrt{q} = e^{-\lambda} + e^{\lambda}$$

$e^{\lambda \#\{\text{signed loops}\}}$

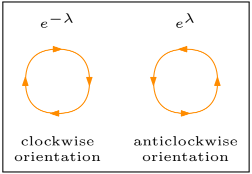
$e^{\frac{\lambda}{2} \#\{\text{signed splits}\}}$

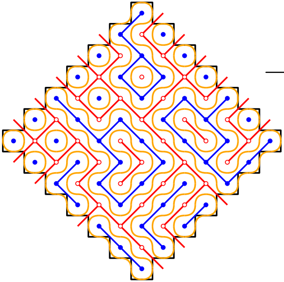
$e^{\frac{\lambda}{2\pi}\{\text{total winding}\}}$

$c^{\#\{\text{type 5 and 6}\}}$

$$c = e^{-\lambda/2} + e^{\lambda/2}$$

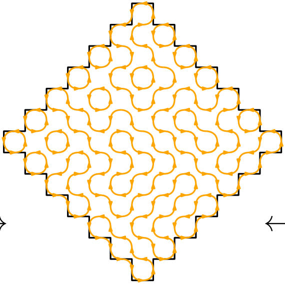
The BKW coupling





$\sqrt{q}^{\#\{\text{inner loops}\}} e^{\lambda \#\{\text{boundary loops}\}}$

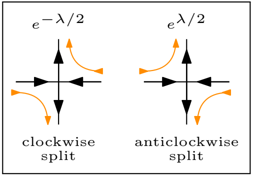
$\sqrt{q} = e^{-\lambda} + e^{\lambda}$

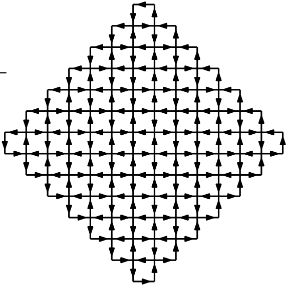


$e^{\lambda \#\{\text{signed loops}\}}$

$e^{\frac{\lambda}{2} \#\{\text{signed splits}\}}$

$e^{\frac{\lambda}{2\pi}\{\text{total winding}\}}$





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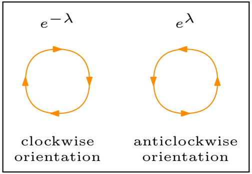
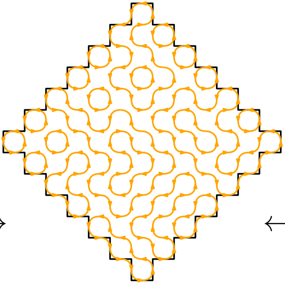
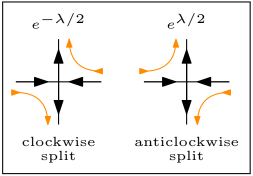
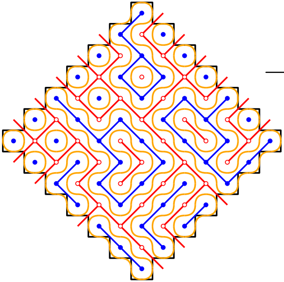
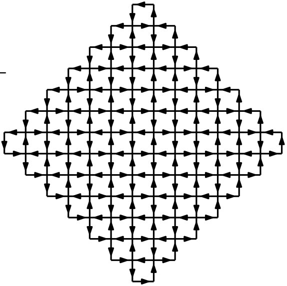
The BKW coupling











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$e^{\frac{\lambda}{2\pi}\{\text{total winding}\}}$

$\sqrt{q}^{\#\{\text{inner loops}\}} e^{\lambda \#\{\text{boundary loops}\}}$

$c^{\#\{\text{type 5 and 6}\}}$

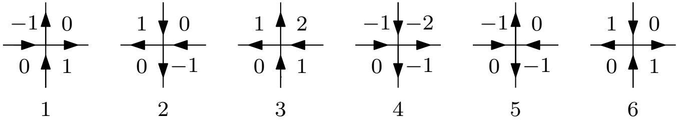
$\sqrt{q} = e^{-\lambda} + e^{\lambda}$

[Glazman–Peled 2019]

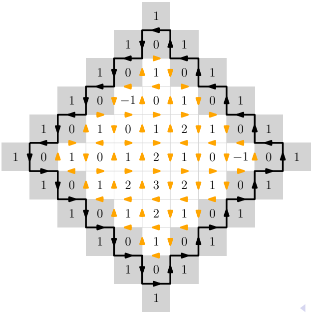
$c = e^{-\lambda/2} + e^{\lambda/2}$

The height function

- A six-vertex config is the **gradient** of a height function:

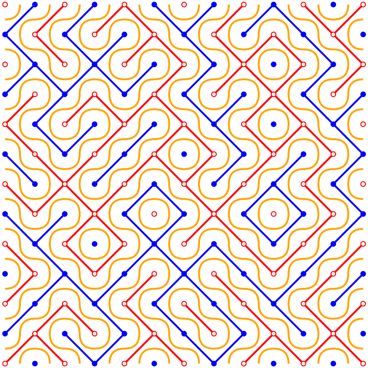


- The height function is defined up to a global additive constant



Proof of discontinuity by contradiction

- Fix $q > 4$ and consider the random-cluster model.



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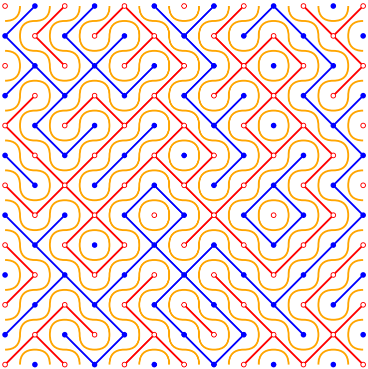
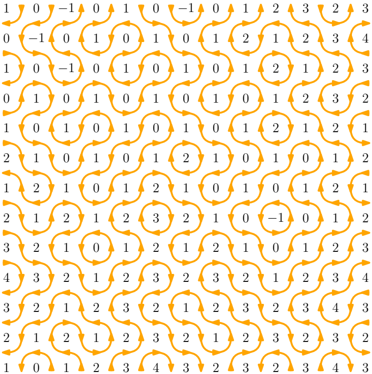


$\longrightarrow$



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- Fix $q > 4$ and consider the random-cluster model.
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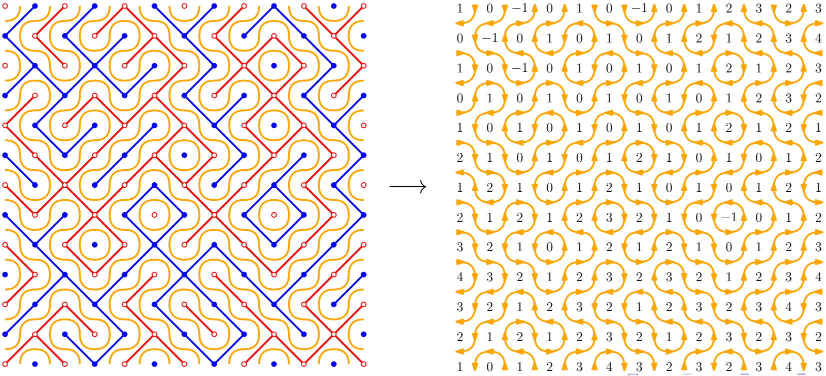

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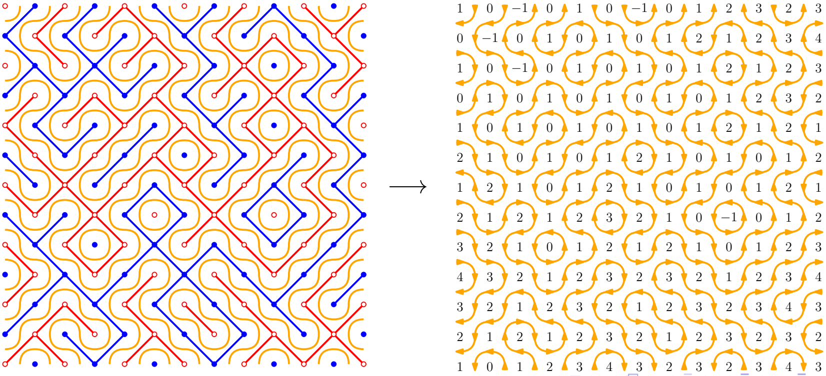

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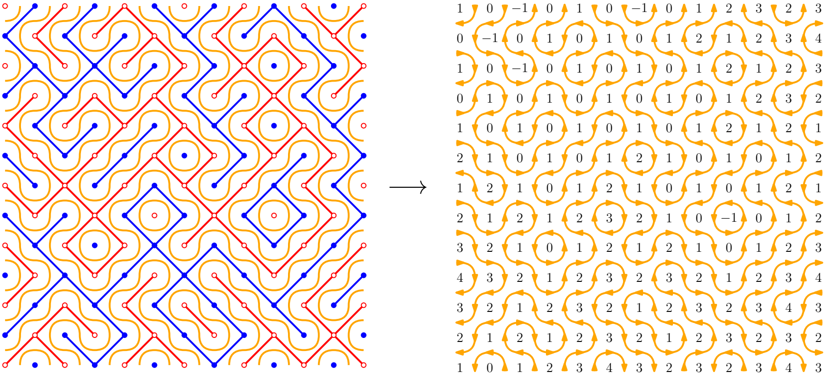
- Every vertex is surrounded by **infinitely many loops**.



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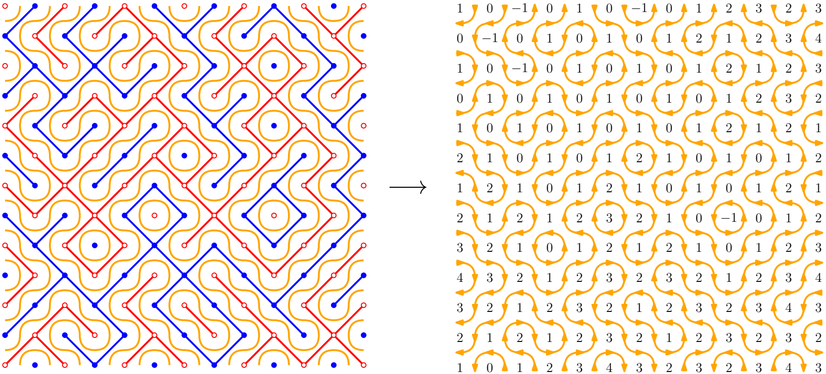

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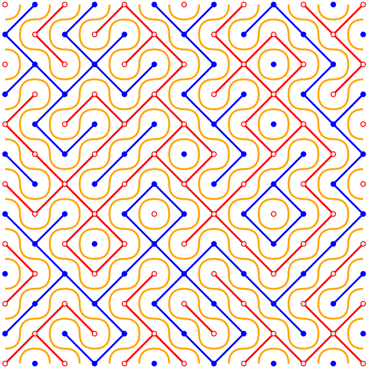

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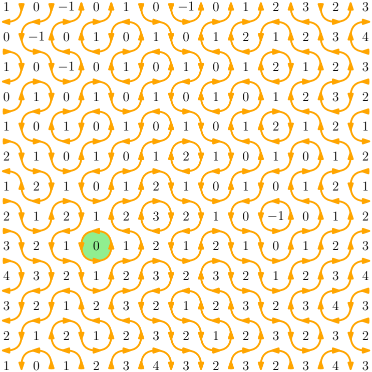

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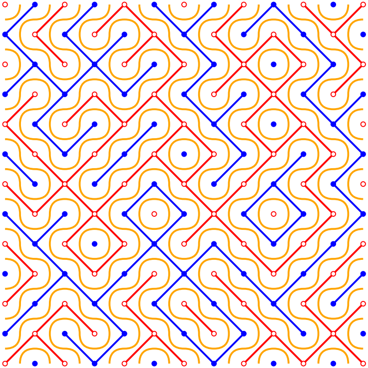

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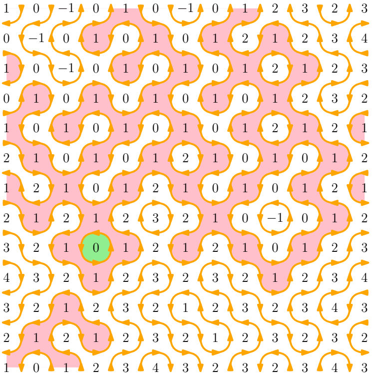

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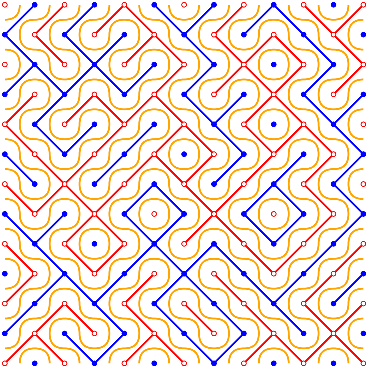

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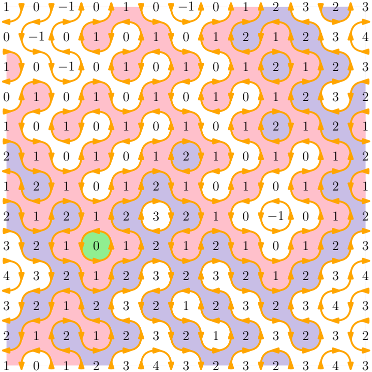

Thank you!