Characterization of the Gaussian free field.

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joint w/ Nathanael Berestycki (U. Vienna)
and
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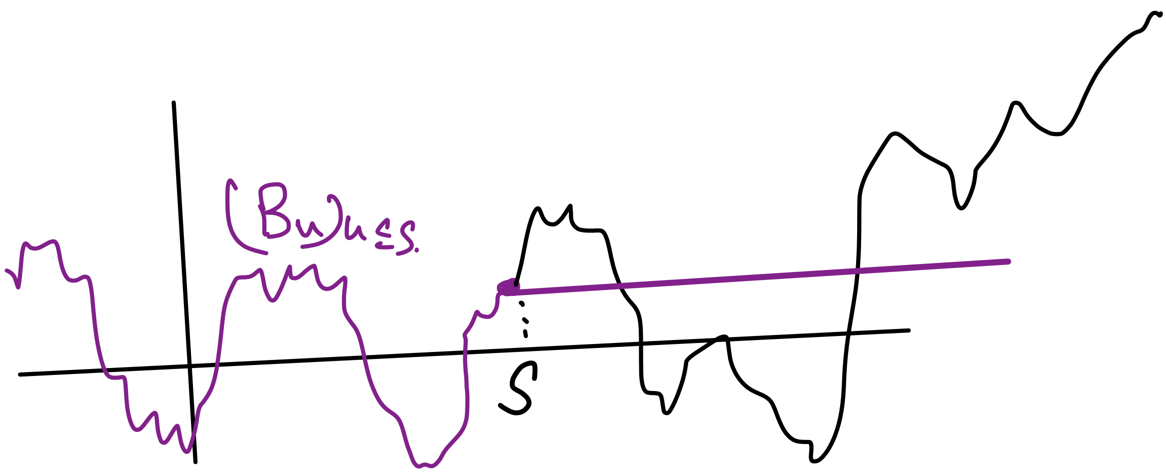
- Brownian motion is, by now, a classical object in probability theory.

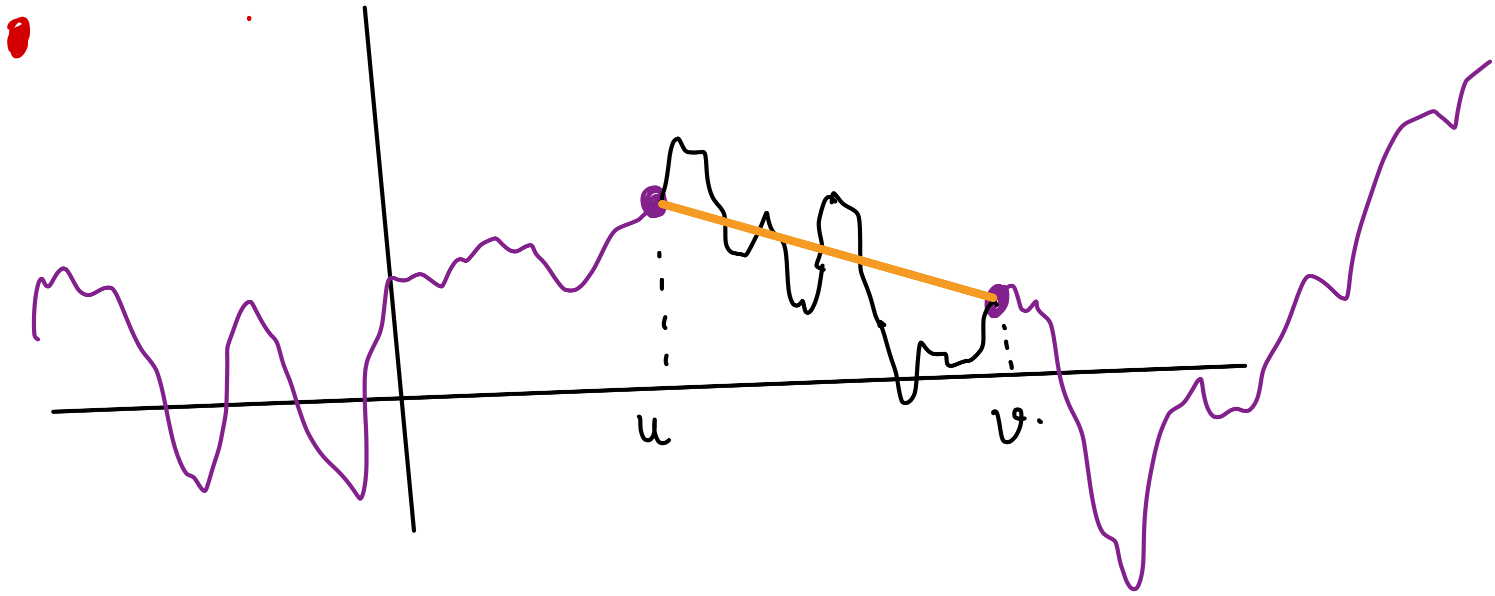
- Universality: Many natural stochastic processes scale to Brownian motion. (Donsker's invariance principle).

- Markov property (one-sided).

$(B_{t-s})_{t>s}$ conditioned on $(B_u)_{u\leq s}$.

$$\overset{(d)}{=} \quad B_s + (\tilde{B}_t)_{t\geq 0}.$$





Law of $(B_x)_{u \le x \le v}$ given $(B_t)_{t \le u,\ t \ge v}$

is given by.

$$\underbrace{\text{Linear interpolation}}_{\text{Harmonic extension.}} + \underbrace{\text{independent Brownian Bridge. between } [u,v]}_{\text{Brownian motion conditioned to be } 0 \text{ at } u \text{ and } v.}$$

and independent of $(B_t)_{t \le u,\ t \ge v}$.

- Scaling property:

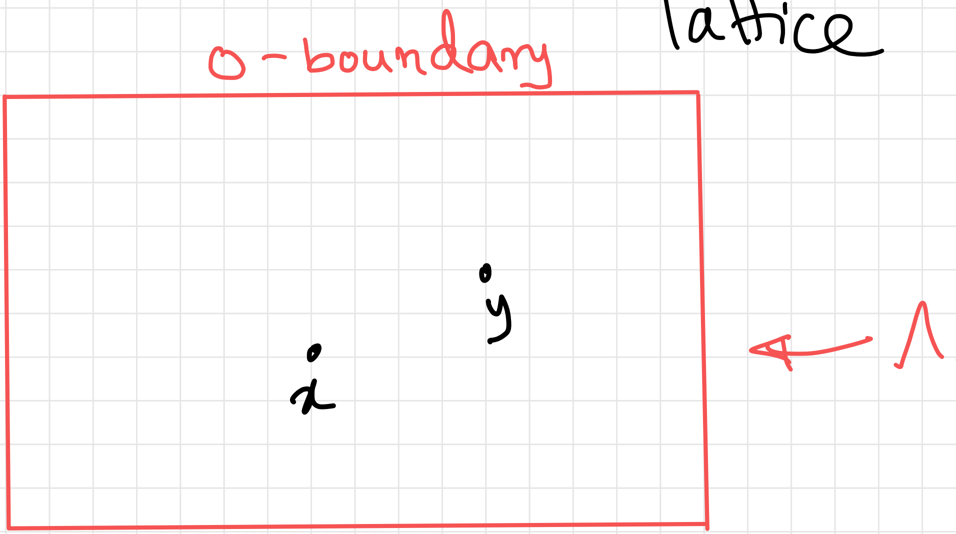
$$\boxed{\frac{1}{\sqrt{c}}\left(B_{ct}\right)_{t\geq 0} \overset{(d)}{=} \left(B_t\right)_{t\geq 0}.}$$

2-D analogue of Brownian motion??

(A "canonical" Gaussian process in $\mathbb{R}^2$, $(h_x)_{x\in\mathbb{R}^2}$, which has a natural "Markov" property and "conformal symmetries"

"Discrete version"

In the 2d square lattice



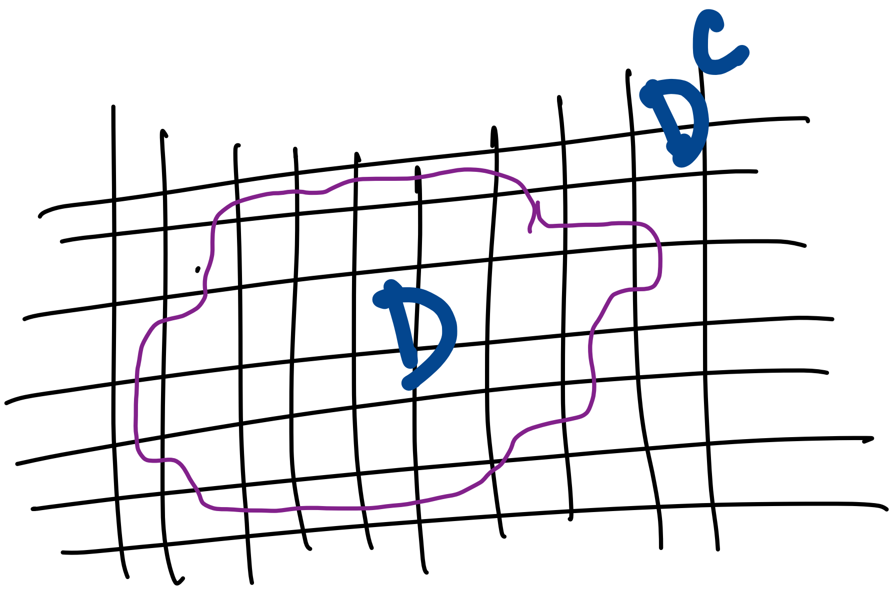
$(h_x)_{x\in\Lambda} \sim$ Multivariate Gaussian.

$\mathbb{E}(h_x)=0$, $\quad \mathrm{Cov}(h_x, h_y) = G^{\Lambda}(x,y).$

$G^{\Lambda}(x,y)$ = Green's function in Λ.

- $G^{\Lambda}(x,y)$: Expected # visits to y by a Random walk started at x before exiting Λ.

Has a natural Markov property.

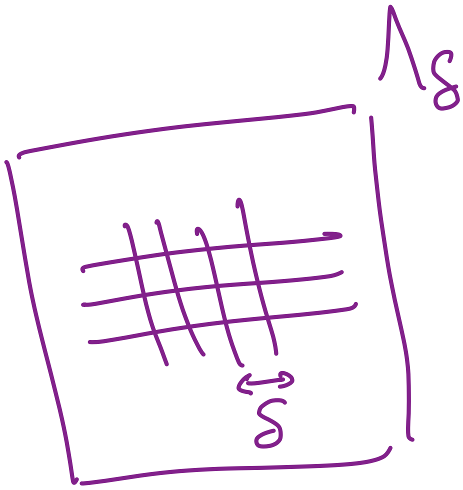


$(h_x)_{x \in D}$ conditioned on $(h_x)_{x \in D^c}$

(d) $\overset{(d)}{=}$ harmonic extension of $(h_x)_{x \in D^c}$

+ independent Gaussian free field in D.

Scaling limit?



$$G^{\Lambda_\delta}(x,x) \simeq \log(1/\delta) \to \infty.$$

Limit does not exist as Random functions!

Discrete GFF

But:

$$\lim_{\delta\to 0} \sum_{v\in D} f^{\delta}(v)\, h^{\delta}(v)\, \delta^2 .$$

$$= N\left(0, \int_{\Lambda} f(x)\, G^{\Lambda}(x,y)\, f(y)\, dx\, dy\right)$$

Continuum 2D-GFF, $D \subseteq \mathbb{C}$, simply conn.

$C_c^\infty(D)$: smooth compactly supported functions in D

Def: $(h_\phi)_{\phi \in C_c^\infty(D)}$.

(stochastic process indexed by smooth functions).

(Endowed with product topology).

(i.e. specified by joint law of
$(h^D_{\phi_1}, h^D_{\phi_2}, h^D_{\phi_3}, \ldots, h^D_{\phi_K})$)

with, $\mathbb{E}(h^D_\phi) = 0 \quad \forall \phi \in C_c^\infty(D)$.

$$\underbrace{\mathrm{Cov}\left(h^D_{\phi_i}, h^D_{\phi_j}\right)}_{\text{Enough to define this by Gaussianity.}} = \int_{D\times D} \phi_i(x) G^D(x,y) \phi_j(y)\,dx\,dy$$

Alternate definition in $H^1(D)$.

$H^1(D)$: Completion of the space of smooth compactly supported function w.r.t. the inner product

$$\langle f,g\rangle_\Delta = \int_D \nabla f\cdot\nabla g \underset{\substack{\uparrow \\ \text{Gauss} \\ \text{Green.}}}{=} -\int_D f\,\Delta g$$

Then take $\alpha_n \overset{iid}{\sim} N(0,1)$ and set

$$\boxed{h = \sum_{n=1}^{\infty} \alpha_n e_n}$$

(e_n: orthonormal basis of eigenfunctions of $-\Delta$).

exists a.s. in H^1.

Γ^D: Law of GFF.

Sometimes we denote

$$h_\phi = (h, \phi)$$

"Think integration"

Properties

- **Conformal invariance (CI).**

Let $f: D \mapsto D'$ be conformal

then $\boxed{\Gamma^D = \Gamma^{D'} \circ f}$

where $\Gamma^{D'} \circ f$ is the law of the stoch. Process

$= (h^D, \phi)$

$$\left(h^{D'}, \; (\phi \circ f^{-1}) \left| (f^{-1})' \right|^2 \right)$$

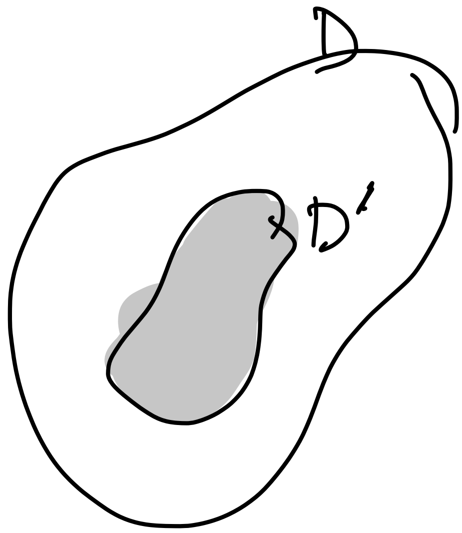
$\phi \in C_c^\infty(D)$

- **(zero/Dirichlet boundary) (DB)**

If ϕ_n hass support $\to \partial D$

and $\phi_n \xrightarrow[H^{-1}(D)]{} 0$ then $(h, \phi_n) \to 0$

• Domain Markov property (DMP).



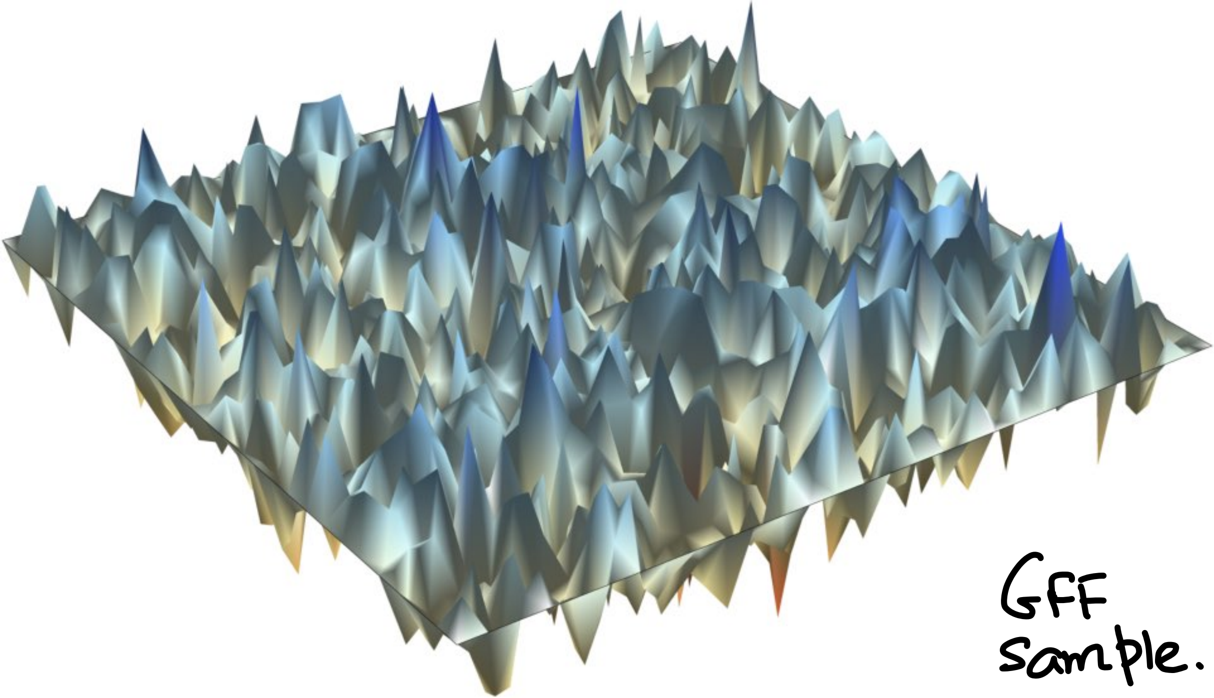
$$h^D = h_D^{D'} + \phi_D^{D'}$$

- $h_D^{D'}$ is independent of h^D

- $\left(h_D^{D'}, \phi\right)_{\phi \in C_c^\infty(D')}$ has law $\Gamma^{D'}$ (GFF on D').

- $\phi_D^{D'}$ is harmonic in D'

($\phi_D^{D'}$ is a stoch. process in $\mathbb{R}^D$ with $(\phi_D^{D'}, \phi)$ is the same as integrating against a harmonic function).



GFF
sample.

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- GFF is a canonical object.
  - scaling limits of many natural stat. physics models:

    e.g: Dimer model, six-vertex model (delocalized phase), Double Ising. etc.

  - Key "perturbation" of harmonic function used in construction of Liouville quantum gravity.

- Random matrix theory.

ETC.

Qn: Let $(\Gamma^D)_{D\subset\mathbb{C}}$ be a family of stochastic processes indexed by $C_c^\infty(D)$. Then does the 3 properties.

conformal inv. $\oplus$ Domain Markov $\oplus$ Dirichlet boundary

$\implies$ Γ^D is GFF on D?

Let h^D: sample from Γ^D.

Thm 1 (Berestycki, Powell, R., '18)

YES if $\mathbb{E}((h^D,\phi)^4) < \infty$ $\forall \phi \in C_c^\infty(D)$

Thm 2 (Berestycki, Powell, R., '20)

YES if $\exists \varepsilon > 0$, $\mathbb{E}((h^D,\phi)^{1+\varepsilon}) < \infty$, $\forall \phi$

- <u>Remarks:</u> Conformal Invariance is not enough.

→ CLE_κ Nesting field (Miller, Watson, Wilson '14)

(Not even Gaussian).

→ Planar Ising magnetization field.

Future work: Characterize fractional Gaussian fields in $\mathbb{R}^d$

$$FGF_s(\mathbb{R}^d) = (-\Delta)^{-s/2} W.$$

W: space time white noise.

$s=1 \longrightarrow$ Gaussian free field.

$s \in (0,1) \approx$ long range GFF with Brownian motion replaced by $2s$-Levy process.

$\boxed{\text{Sketch of Proof (Thm 1)}}$

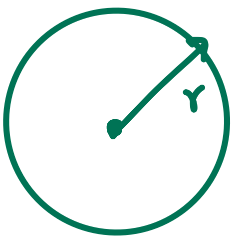
- Step 1: Show h^D is Gaussian
- Step 2: Show covariance is Green's function.

Key tool: Circle average process

$$(h^D, \rho_r)$$

(Lemma: This exists under CI + PMP).

Where $(f, \rho_r) = \frac{1}{2\pi r}\int_{\partial B_r} f(x)\,dx$



(ρ_r is not a smooth function, so, this definition needs justification!)

Sketch of Proof (Thm 1)

Proof of Step 2 (Two point function).

(Technical Step): $\exists$ a Covariance kernel

$$\boxed{K^D(z_1,z_2) := \lim_{\varepsilon\to 0} \mathbb{E}\left(h_\varepsilon^D(z_1), h_\varepsilon^D(z_2)\right)}$$

Where $h_\varepsilon(z) = (h, \rho_\varepsilon)$

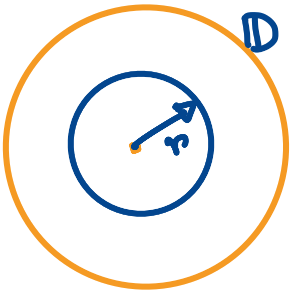
• If $f: D \mapsto D'$ is conformal.

$$K^{f(D)}(f(z_1), f(z_2)) = K(z_1, z_2).$$

$\mathbb{D}$: unit disc

• $K^{\mathbb{D}}(0, y) = -a \log|y|, \; y \in \mathbb{D}.$

Proof: Let $f(r) = \mathbb{E}\left[\left(h_r^{\mathbb{D}}(0)\right)^2\right]$



$$= \mathbb{E}\left(\left(h^{\mathbb{D}}, \rho_r\right)^2\right)$$

Conf. Inv. + Domain Markov.

$$\Rightarrow \boxed{f(rs) = f(r) + f(s)}$$

$r, s < 1$

- f is continuous (technical estimates)
- $f(1) = 0$.

$$\Rightarrow \quad f(s) = -a \log(s). \qquad s < 1$$

but $f(r) = \int_{\partial B_r} K^{\mathbb{D}}(0, w)\, \rho_r(w)\, dw$

Conf. Inv of K $\Rightarrow$ $K^{\mathbb{D}}(0, w) = f(|w|)$
$$= -a \log|w| \qquad \blacksquare$$

$\boxed{\text{Sketch of Proof (Thm 1)}}$

- Take $D = \mathbb{D}$. (unit disc).
- Take $B_t = \left(h^{\mathbb{D}}, \rho_{e^{-t}}\right)_{t \geq 0}$

$$\boxed{\text{Scale invariance} \Rightarrow \text{stationarity}}$$

$\forall c$ $\quad (B_{t+c})_{t\in\mathbb{R}} \stackrel{d}{=} (B_t)_{t\in\mathbb{R}}.$

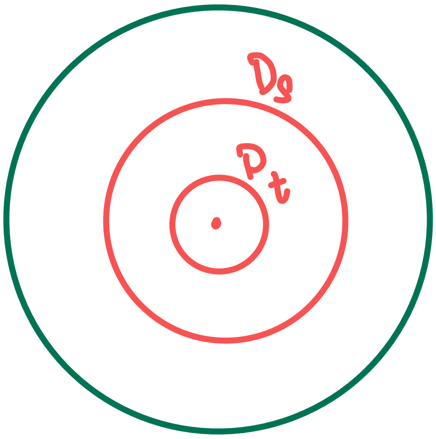
- B_t is a martingale: Fix $s<t$.
  with independent increment.

$D_u = e^{-u}\mathbb{D}$

$$\mathbb{E}\left(B_t - B_s \mid (\mathcal{F}_u)_{u\leq s}\right)$$

$$= \mathbb{E}\left(\underbrace{\left(h^{D_s}_{\mathbb{D}}, \rho_{e^{-t}}\right)}_{\text{indpt of } \mathcal{F}_s} + \left(\varphi^{D_s}_{\mathbb{D}}, \rho_{e^{-t}}\right)\right.$$

cancels by harmonicity



$$- \underbrace{\left(h_{ID}^{D_t}, f_{e-t}\right)}_{\overset{\shortparallel}{0}} - \left(\left(\varphi_{ID}^{D_t}, f_{e-t}\right) \middle| \mathcal{Y}_S \right)$$

$$= \mathbb{E}\left(\left(h_{ID}^{D_S}, f_{e-t} \right) \right)$$

$$= 0$$

$$?? \ \frac{1}{\sqrt{c}}\left(B_{ct}\right) \overset{(d)}{=} B_t$$

Showing B_t is cont. is hard part. (rules out centered Poisson Process etc.)

- No scaling relation.
- Need 4th moment here to estimate the

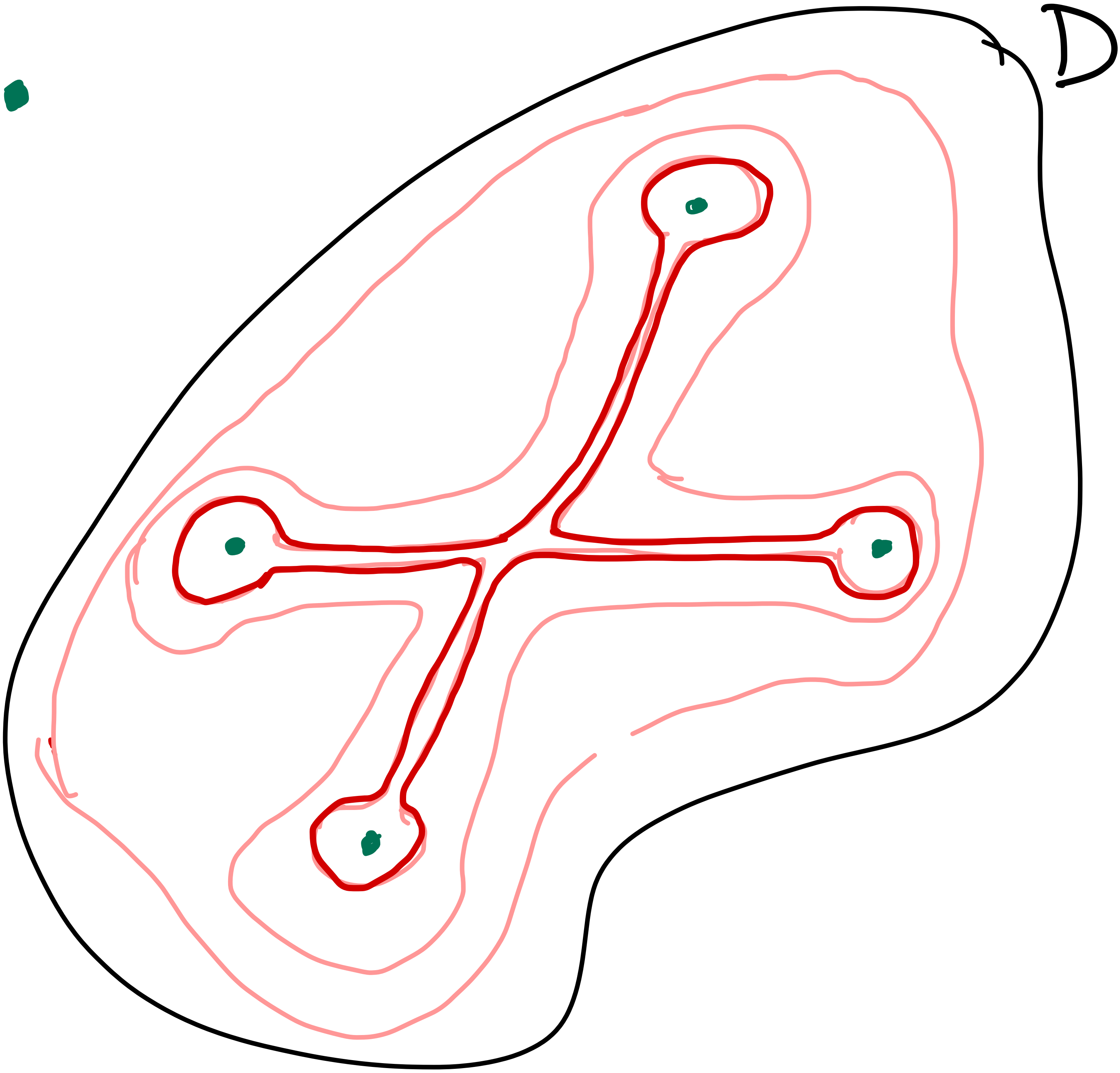
harmonic functions and get

$$\mathbb{E}\left[\left(B_t - B_{t+\varepsilon}\right)^4\right] \leq C\varepsilon^{1+\eta}.$$

$\Rightarrow B_t$ is a Brownian motion

[Note $\mathbb{E}\left[\left(B_t - B_{t+\varepsilon}\right)^2\right] = c\,\varepsilon$, NOT enough!)

- To show $\boxed{\text{joint Gaussianity of } h_\varepsilon(z_1), h_\varepsilon(z_2), \ldots, h_\varepsilon(z_k)}$.
- Need to extend f_ε to a notion of "harmonic" average (can be defined directly OR by conformal invariance).



- Find a sequence of domains approximating the joint circle average.

Proof of Theorem 2.
(Lowering the moments).

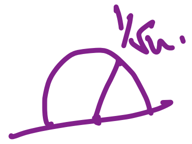
(A) Show Gaussianity of single circle averages by showing ∃ a.s. continuous modification.

(B) Deduce existence of 4th moments of (h^D, ϕ) from this.

Part (A): Take the upper half plane $\mathbb{H}$. Consider the measure p_u

$$(\phi, p_u) = \sqrt{u} \int_0^{\pi} \sin(\theta)\, \phi\left(\frac{e^{i\theta}}{\sqrt{u}}\right) d\theta$$

p_u: (Ito excursion measure)



- Supported on $\partial\left(\frac{1}{\sqrt{u}}\mathbb{D} \cap \mathbb{H}\right)$
- Does not have total mass 1.

- We study "sine averages".

$$\boxed{Y_u := (h^{\mathbb{H}}, p_u), \quad u > 0.}$$

Ito excursion measure

$$\mathcal{N} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \mathbb{P}_{i\varepsilon}$$

$\mathbb{P}_{i\varepsilon}$: Law of Brownian Motion started at $i\varepsilon$ killed on reaching $\mathbb{R}$.

Lemma: Mass of excursions leaving $r\mathbb{D} \cap \mathbb{H}$ through (re^{ia}, re^{ib}) is

$$\frac{2}{\pi r} \int_a^b \sin(\theta)\, d\theta.$$

Proof of Theorem 2 (contd).

Recall: $Y_u = (h^{H}, p_u)$, $u > 0$.

Prop: Y_u has a modification which is $\sigma \times$ (standard BM).

Proof: Employing previous ideas

(a) (Scaling). $\forall\, c > 0$

$$\frac{1}{\sqrt{c}}\left(Y_{cu}\right)_{u>0} \stackrel{(d)}{=} \left(Y_u\right)_{u>0}$$

(Hard to prove for circle averages).

(b) $\left(Y_t\right)_{r<t<s}$ conditioned on $\left(Y_u\right)_{u\leq r}$ and $\left(Y_u\right)_{u\geq s}$ is

a "linear interpolation"

+ something independent.

(Hard to prove for circle average).

Theorem (Wesolowski '93):

(a) + (b) + $\mathbb{E}(Y^2(u)) < \infty \ \forall u$.

$\Rightarrow$ $Y = \sigma \times$ Brownian Motion).

- With some work we can show Wesolowski assuming $\boxed{\mathbb{E}(|Y(u)|^{\xi}) < \infty \text{ for some } \xi > 0}$

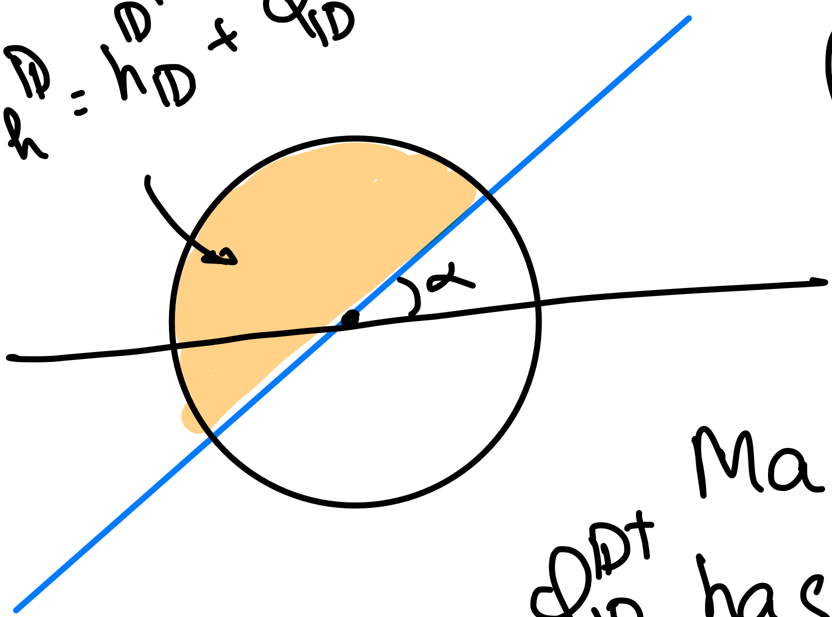
(seems to be new!).

Proof of Theorem 2 (contd).

- Bootstrap the sine-averages to get circle-average.

- Idea: "random rotation" and "unif rotation of sine average" = circle average.

$h^{\mathbb{D}} = h_{\mathbb{D}}^{\mathbb{D}^+} + \phi_{\mathbb{D}}^{\mathbb{D}^+}$



rotation

$\left(h_{\mathbb{D}}^{\mathbb{D}^+}, P_u^{\alpha}\right) = BM.$

Make sure $\phi_{\mathbb{D}}^{\mathbb{D}^+}$ has "little effect"!